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Finite Compression of Derived Quot Presentations via Homogeneity and Rank Propagation

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Abstract

The Derived Quot schemes retain the obstructive information which is lost in a classical Quot scheme. Their finite stretch construction is typically written using products of Grassmannians, multilinear Hom-spaces, Maurer–Cartan equation, and quotient by products of general linear groups. In this paper, we give a finite criterion under which the same bounded derived Quot data is captured by a shorter Grassmannian stretch taking into account at once both the homogeneity span of each operation and the rank propagation of admissible submodules. This criterion, called rank-sealed span audit, is based on filtering the differential graded Lie algebra whose elements are

$$\mathrm{Hom}_0(R_+^{\otimes k} \otimes N_{\geq a}, N_s) \oplus \mathrm{Hom}_0(R_+^{\otimes (k-1)} \otimes N_{\geq a}, M_s),$$

by the difference of homogeneity degrees of input and output and detecting its acyclic coordinate pairs by means of the associated spectral sequence; after that one removes the Koszul dual coordinates that do not contribute to the tangent cohomology up to degree k. The resulting affine dg Lie algebra is then further reduced by imposing on it the rank condition $\ker(N_s \to M_s) = 0$ and the generation condition $\mathrm{coker}(R_{s-s'} \otimes N_{s'} \to N_s) = 0$ that allows forming a quotient over the shorter base $\prod_{s=a}^{b} \mathrm{Gr}(h(s), M_s)$. We prove that the reduced presentation has the same classical admissible points and the same tangent cohomology up to degree k as the corresponding finite stretch derived Quot model; the finite-generation argument gives a weakly equivalent dg-manifold of finite type. The construction provides an algebraic record that is finite, a concrete audit process, and a theorem-based criterion that determines whether a bounded derived Quot presentation is already known through a brief period of homogeneity.

Keywords: derived Quot schemes; dg manifolds; Maurer–Cartan equations; Grassmannian quotients; obstruction theory; invariant theory; cotangent complexes

1. Motivation for algebraicity and a finite criterion

Classical Quot schemes solve a representability problem for coherent quotients of a projective scheme. A given projective variety and a coherent sheaf provide inputs for Grothendieck's construction of a projective scheme classifying quotients with prescribed Hilbert polynomial [1]. Standard constructions of modern projective algebraic geometry supply the coherent-sheaf, Hilbert-polynomial, and projective-embedding contexts of this construction [2]. Finite generation, regularity, and the possibility to express module compatibility in terms of algebraic equations on a finite product of Grassmannians enter the picture. The machinery of geometric invariant theory constructs quotients under linearized group actions [3]. Modern approaches to moduli of sheaves incorporate Grassmannian embeddings and tautological bundles within this invariant-theoretic setup [4]. Simpson's construction provides yet another model for the construction of projective moduli spaces via geometric invariant theory [5]. Derived Quot construction asks for more than a space of closed points. It asks for a moduli object whose structure sheaf captures the non-transversality of the classical equations.

Deformation theory explains the necessity of this additional structure. Schlessinger's functorial criteria characterize the cases where an infinitesimal deformation problem admits a controlled hull [6]. Artin's algebraization theorem provides the conditions under which the formal deformation data come from an algebraic moduli object [7]. The first treatment of the cotangent complex provides the organization of tangent and obstruction spaces in terms of homology [8]. The second volume develops the deformation-theoretic implications of the cotangent complex [9]. Perfect obstruction theories use such complexes to provide virtual fundamental classes for classical moduli spaces [10]. Examples of deformation–obstruction complexes for sheaves show the power of this approach in enumerative geometry [11]. Derived algebraic geometry incorporates the higher structure directly into the definition of the moduli object. Homotopical algebraic geometry provides a systematic model-categorical framework for this purpose [12]. Its theory of geometric stacks extends the construction to the case of derived moduli problems [13]. Later surveys explain the intrinsic meaning of these objects from the point of view of non-transversal intersections and obstructions [14]. Shifted symplectic structures show that derived moduli stacks can carry canonical geometric forms beyond the classical case [15]. Derived Quot schemes thus fit into a wider family of objects featuring tangent complexes, obstruction classes, and homotopy-invariant quotients.

The derived Quot construction was introduced by Ciocan-Fontanine and Kapranov by substituting ordinary finite-stretch submodules by A_∞-submodules and modeling the resulting conditions by a differential graded Lie algebra [16]. The relevant differential graded Lie algebra is finite-dimensional over a chosen homogeneity interval and is translated via Koszul duality to a commutative differential graded algebra. Operadic Koszul duality provides the algebraic foundations of this translation [17]. Hinich's descent theory connects dg Lie algebras with homotopy-invariant deformation problems [18]. Manetti develops the Maurer–Cartan and differential-graded-Lie-theoretic approach to deformation functors [19]. Pridham provides yet another description of derived deformation theory in these terms [20]. The construction also interacts with invariant theory since the vector spaces representing the graded pieces have change-of-basis symmetries. Invariant theory distinguishes algebraic and geometric quotients of linear group actions [21]. The machinery of geometric invariant theory provides the stable locus conditions under which orbit separation becomes geometric [3].

Finiteness of representation as dg manifolds is a stronger assertion than representation as derived schemes since dg manifolds carry a smooth quasi-projective classical part and an almost free differential graded structure sheaf generated by coherent modules in non-positive degrees. Finiteness of representation of derived Quot objects as dg manifolds under coherent generation is provided by the finite-type dg-manifold framework [22]. Application of this finite-type approach to the construction of derived Quots provides the finite-type dg-manifold models for finite-stretch quotient stacks [23]. The proof divides the difficulties associated with projective transition maps from those of affine nature necessary for computations with limits of dg schemes. It uses an injectivity interval $[a, b]$, affine quotient replacements, geometric quotients on maximal-rank loci, and a Postnikov-type stabilization procedure. The current paper isolates the formal ingredients of the above proof in order to provide a finite recognition criterion for bounded presentations.

The finite criterion is stated for a smooth connected projective variety $(X, \mathcal{O}_X(1))$ over $\mathbb{C}$, a finitely generated graded module M_* over the homogeneous coordinate ring, and a Hilbert polynomial h specifying the dimensions of the graded subspaces N_s. The finite criterion identifies when the bounded derived Quot presentation over a long interval $[a, t]$ is obtained from the shorter Grassmannian interval $[a, b]$. The two required verifications are the spectral acyclicity of long-span Hom operations and rank propagation of admissible graded pieces. The construction does not approximate the derived Quot object and does not drop the obstruction terms via a crude degree cut-off. Instead it filters the finite Maurer–Cartan algebra by homogeneity span and drops only acyclic coordinate pairs invisible to tangent cohomology up to degree k.

The construction is performed on a finite symbolic corpus. It consists of graded vector spaces M_s, vector spaces N_s with $\dim_{\mathbb{C}} N_s = h(s)$, products of Grassmannians $\mathrm{Gr}(h(s), M_s)$, Hom spaces defining A_∞-module operations and morphisms, group actions by $\mathrm{GL}(h(s), \mathbb{C})$, and the rank conditions representing injectivity and generation. These objects are not measurements of observations; they are the finite algebraic input on which the derived Quot equations, the span filtration, and the quotient construction are being performed. The resulting bounded dg presentation keeps the classical admissible locus and does not change the tangent cohomology in degrees $[0, k]$.

2. Quot geometry before compression

The classical and derived constructions both start from the same graded input data. The coordinate ring of $(X, \mathcal{O}_X(1))$ is

$$R_* = \bigoplus_{t\in\mathbb{Z}} R_t, \qquad R_t = \Gamma(X, \mathcal{O}_X(t)),$$

and $R_+ = \bigoplus_{t\geq 1} R_t$ is the positive-degree irrelevant part. The module $M_* = \bigoplus_{t\in\mathbb{Z}} M_t$ is finitely generated over R_*. The construction of the Picard scheme due to Kleiman contains an account of the methods used for deriving uniform finite bounds on the regularity [24]. The treatment of the Castelnuovo–Mumford regularity of graded modules in the language of commutative algebra may be found in Eisenbud's book [25]. Once a is chosen sufficiently large, the submodules in question are spanned by finite graded pieces and satisfy $\dim_{\mathbb{C}} N_s = h(s)$ for all $s \geq a$. For a finite interval $[a, t]$, let

$$N_{a,t} = \bigoplus_{s=a}^{t} N_s, \qquad M_{a,t} = \bigoplus_{s=a}^{t} M_s.$$

The classical parameter space before imposing module equations is the product

$$\mathrm{Gr}_{[a,t]} = \prod_{s=a}^{t} \mathrm{Gr}(h(s), M_s).$$

Each factor remembers only the inclusion of a vector space of dimension $h(s)$ into M_s; the multiplication constraints imposed by R_+ have still to be enforced.

The coordinates make clear the position of the current construction in the framework of algebraic geometry. At the same time, the table points out the contribution made by the paper, which does not add any new moduli functors but provides

Table 1: Literature coordinates.

Theme	Established role	Use in this paper
Classical Quot representability	Grothendieck constructs projective parameter spaces for quotients with fixed Hilbert polynomial [1].	Supplies the finite Grassmannian interval $\mathrm{Gr}_{[a,t]}$ and the regularity dependence on a.
Geometric invariant theory	Stable loci are handled through geometric invariant theory [3]. Algebraic and geometric quotients are distinguished in invariant theory [21].	Separates algebraic quotient, partial quotient, and geometric quotient in the compressed construction.
Cotangent and deformation theory	Cotangent complexes organize tangent and obstruction data [8]. Deformation functors satisfy representability criteria [6]. Dg Lie algebras encode homotopy-invariant deformation equations [18].	Determines which coordinate directions can be removed without changing tangent cohomology.
Derived Quot theory	Finite-stretch A_∞-submodules model the derived Quot functor [16]. Finite-type derived Quot models use the resulting dg algebra [23].	Provides the Hom-space algebra on which the span audit is performed.
Finite-type dg manifolds	Coherent cohomology can be replaced by locally free dg generators [22]. Derived geometry supplies the ambient homotopical interpretation [14].	Converts the compressed dg scheme into a finite-type dg manifold after bounded equivalence is proved.

an audit for the finite hom-space algebra which determines when the smaller Grassmannian base already determines the bounded derived construction. The coordinates are purposely listed by their role in mathematics since the construction uses all five of them simultaneously, and not in strict chronological order.

The finite homogeneity window is illustrated in Figure 1. The small interval $[a, b]$ includes the explicit rank condition, whereas the degrees above b are obtained by R_+-multiplication. Such distinction is important for the construction since the base of the resulting quotient is $\mathrm{Gr}_{[a,b]}$ and not $\mathrm{Gr}_{[a,t]}$.

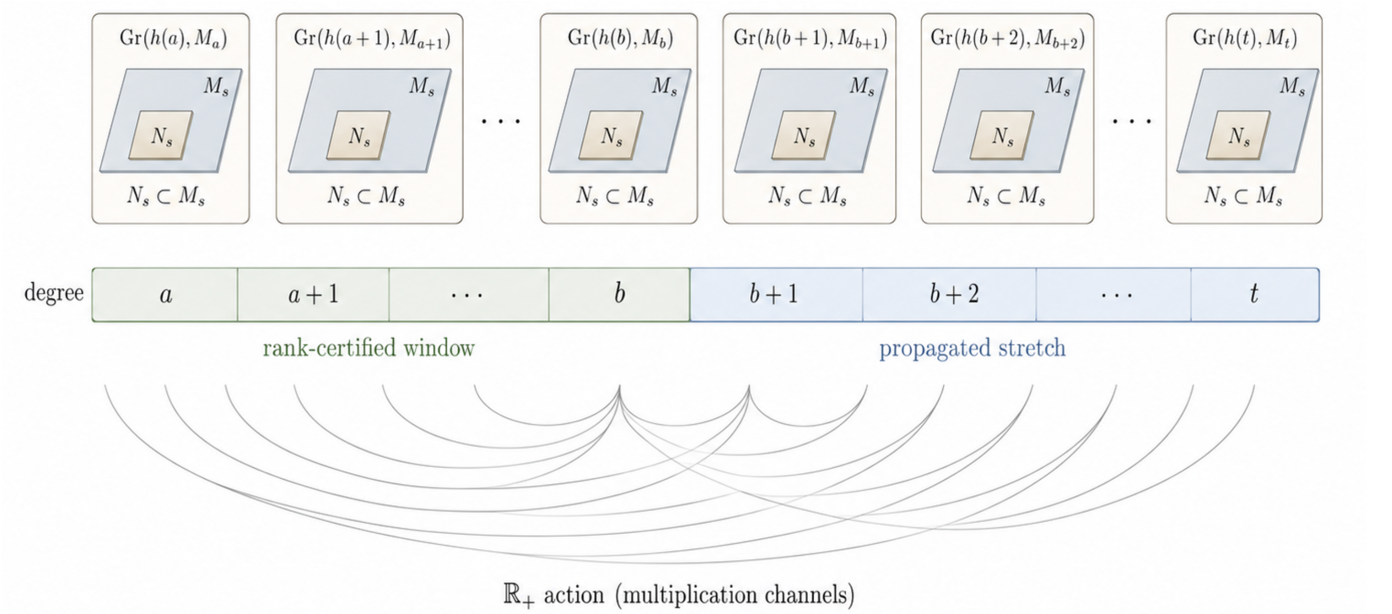

Figure 1: Homogeneity window.

The difference between the short window and the propagated stretch is defined algebraically. Subspaces $N_s \subset M_s$ are chosen along the short Grassmannian base, and higher consistency is verified by means of multiplication maps of lower degrees. Lower channels in the diagram correspond to the process via which generation in the short time window restricts generation of higher graded pieces. In the construction below, the latter channels are not presented as arrows since the relation does not consist of one morphism, but rather of a family of module operations subject to the Maurer–Cartan equation.

The derived finite-stretch algebra encodes the higher consistency conditions of those module operations. For $k \geq 1$ and

$a \leq s \leq t$, set

$$g^k_{\geq a;s} = \mathrm{Hom}_0(R_+^{\otimes k} \otimes N_{\geq a}, N_s) \oplus \mathrm{Hom}_0(R_+^{\otimes(k-1)} \otimes N_{\geq a}, M_s).$$

The first summand consists of operations that go into the candidate submodule. The second summand consists of components of the morphism corresponding to the A_∞-module structure of $N_{\geq a}$ in the fixed module $M_{\geq a}$. Taking all output degrees into account leads to the sum

$$g^*_{a,t} = \bigoplus_{s=a}^{t} \bigoplus_{k \geq 1} g^k_{\geq a;s}.$$

Composition of multilinear operations leads to a graded Lie bracket, while the multiplication on R_+ and the fixed module structure on M_* provide a Maurer–Cartan differential. Therefore, $g^\bullet_{a,t}$ is a finite-dimensional differential graded Lie algebra.

The operation ledger from Figure 2 reflects the two families of Hom's in rows labeled by the output space and the arity. The representation clearly shows that the dg Lie algebra is finite only due to finiteness of the homogeneity interval. At the same time, it clarifies why the morphism part of the construction cannot be dropped: the derived Quot problem is not the deformation of the abstract A_∞-module $N_{\geq a}$, but rather the deformation of its inclusion-like morphism into the fixed module $M_{\geq a}$.

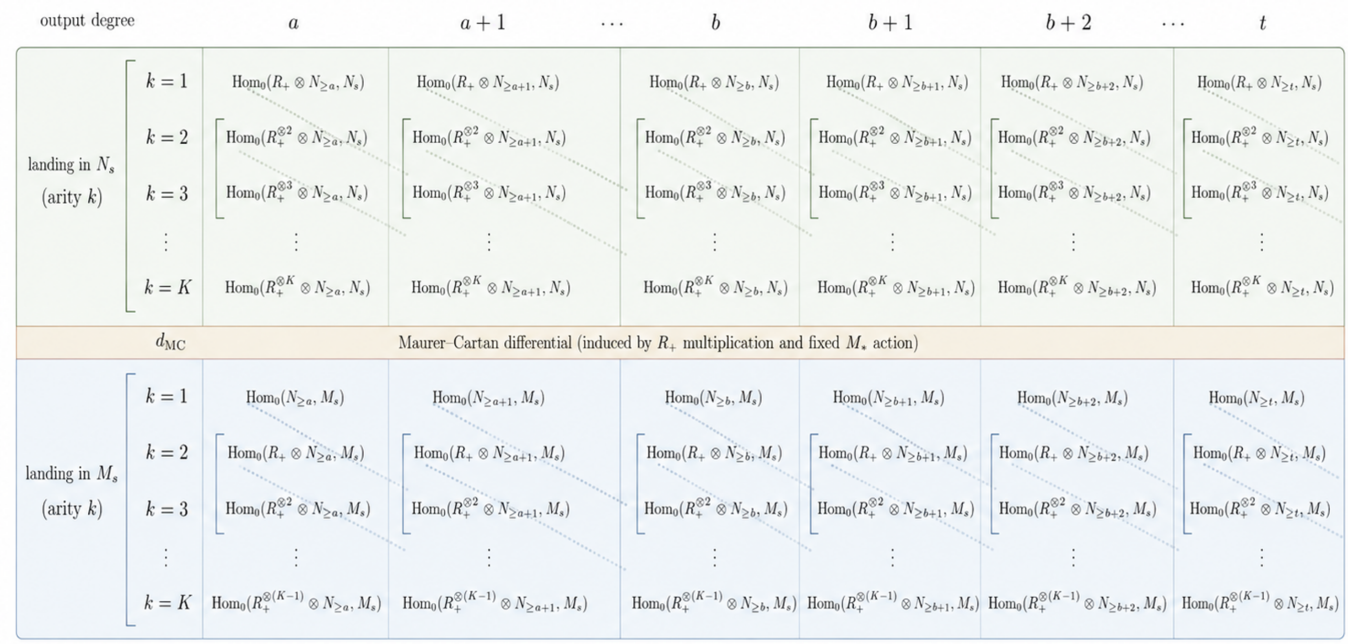

Figure 2: Operation ledger.

In our discussion the ledger is understood as an algebraic object with a finite number of entries and not as a process diagram, where each entry is a vector space of multilinear maps, the diagonal structures encode compositions, and the horizontal division into N_s-valued and M_s-valued entries corresponds to two pieces of the derived Quot equations. In this way we avoid an equation only reading of $g^\bullet_{a,t}$, because Lie algebra structure includes internal deformation and compatibility of modules in the ambient graded module.

The Koszul duality transformation turns $g^\bullet_{a,t}$ into a commutative differential graded algebra

$$A^\bullet_{a,t} = \left(\mathrm{Sym}((g^*_{a,t}[1])^\vee), d_{a,t}\right),$$

with dual to Lie differential and Lie bracket differential. The affine dg scheme $\mathcal{M}_{a,t} = \mathrm{Spec}\, A^\bullet_{a,t}$ describes finite-stretch Maurer-Cartan coordinates prior to rank and quotient restrictions. The change of basis group

$$G_{a,t} = \prod_{s=a}^{t} \mathrm{GL}(h(s), \mathbb{C})$$

acts on the ledger by a change of bases in vector spaces N_s. The locus of rank condition is obtained by imposing injectivity of the maps $N_s \to M_s$ corresponding to a Maurer-Cartan point. The quotient of the space of injective embeddings by $G_{a,t}$ gives the product of the Grassmannians $\mathrm{Gr}_{[a,t]}$. This results in the well-known finite stretch derived Quot presentation [16]. The finite-type dg-manifold realization comes next via coherent replacement [23].

Table 2: Finite algebraic corpus.

Object	Concrete form	Function in the audit
Graded ambient module	$M_{a,t} = \bigoplus_{s=a}^{t} M_s$	Carries the fixed target of all inclusion-like morphisms.
Candidate graded pieces	$N_{a,t} = \bigoplus_{s=a}^{t} N_s$, $\dim N_s = h(s)$	Supplies the variable submodule pieces acted on by basis changes.
Grassmannian base	$\prod_{s=a}^{b} \mathrm{Gr}(h(s), M_s)$	Retains only the short interval where rank is imposed directly.
Hom-space algebra	$g^{k}_{\geq a;s}$ from the two Hom families	Defines the finite Maurer–Cartan equations and their bracket.
Rank tests	$\ker(N_s \to M_s)$ and $\mathrm{coker}(R_{s-s'} \otimes N_{s'} \to N_s)$	Certifies injectivity and generation beyond the short window.
Compression ideal	$\mathcal{I}^{\mathrm{ac}}_{a,b,k,t} \subset A^{\bullet}_{a,t}$	Removes only acyclic long-span coordinate directions.

This fixed body of data defines the mathematical objects which will be used for the rest of the paper. The elements have their specific use in the construction of the derived Quot scheme. The small Grassmannian base contains the classical coordinate information which still shows up after the compression, the algebra of the Hom space gives the higher equations, and the rank tests decide if the higher stretch has been correctly created. This is not a process involving any number sampling or measurement; it uses only the finite inputs themselves.

3. Span audit of the Maurer–Cartan algebra

The key idea is a filtration by span. If a multilinear operation acts on an input of degree q and lands in an output of degree p, its span is $p - q$. On a finite segment $[a, t]$, the span is in the range from 0 to $t - a$. The filtration by span is

$$F^{\ell} g^{\bullet}_{a,t} = \{\text{operations of span at most } \ell\}, \qquad 0 \leq \ell \leq t - a.$$

The composition preserves the span, as the degree travel of a composite operation is equal to the sum of degree travels of operations composing. Therefore,

$$[F^{\ell} g^{\bullet}_{a,t}, F^{m} g^{\bullet}_{a,t}] \subset F^{\ell+m} g^{\bullet}_{a,t}.$$

The Maurer–Cartan differential is defined via the multiplication in R_+ and the fixed action on M_*; hence, it also satisfies the above condition. Thus, there is a spectral sequence for the filtration of dg Lie algebra, and by duality, for the filtration of the coordinate algebra.

The span diagram in Figure 3 distinguishes operations according to their difference between the input and output degrees. The diagonal region stands for operations governed by the short segment, whereas the far region corresponds to more long travel of homogeneity. Point styles instead of colors are used because it is the homological, not the numerical property; a class survives in the tangent range under consideration, whereas an acyclic pair is killed by the differential.

The diagram is essential as it differentiates two concepts which tend to be confused in the finite segment analysis. High homogeneity does not necessarily mean irrelevance, and low homogeneity does not necessarily mean relevance. Relevance is determined by the spectral sequence for the filtration of the dg Lie/Maurer–Cartan algebra, and a retained point corresponds to a surviving class contributing to the bounded tangent cohomology, while a paired point stands for acyclic pairs which may be deleted without altering the local derived structure within the selected range.

The coordinate algebra inherits the filtration via duality. The indecomposable part of $A^{\bullet}_{a,t}$ is $(g^{*}_{a,t}[1])^{\vee}$; thus, the filtration on $g^{\bullet}_{a,t}$ induces the multiplicative filtration on $A^{\bullet}_{a,t}$. There is a spectral sequence

$$E_r^{p,q}(a,t) \Longrightarrow H^{p+q}(A^{\bullet}_{a,t})$$

wherein the span index is recorded in p and the cohomological degree is recorded in $p + q$. At a Maurer–Cartan point, the same filtered construction is applied to the cotangent complex. The class of coordinates is active in the tangent range $[0, k]$ if its cotangent class contributes in degrees $[-k, 0]$. Only those classes may be deleted which belong to an acyclic pair outside that active range.

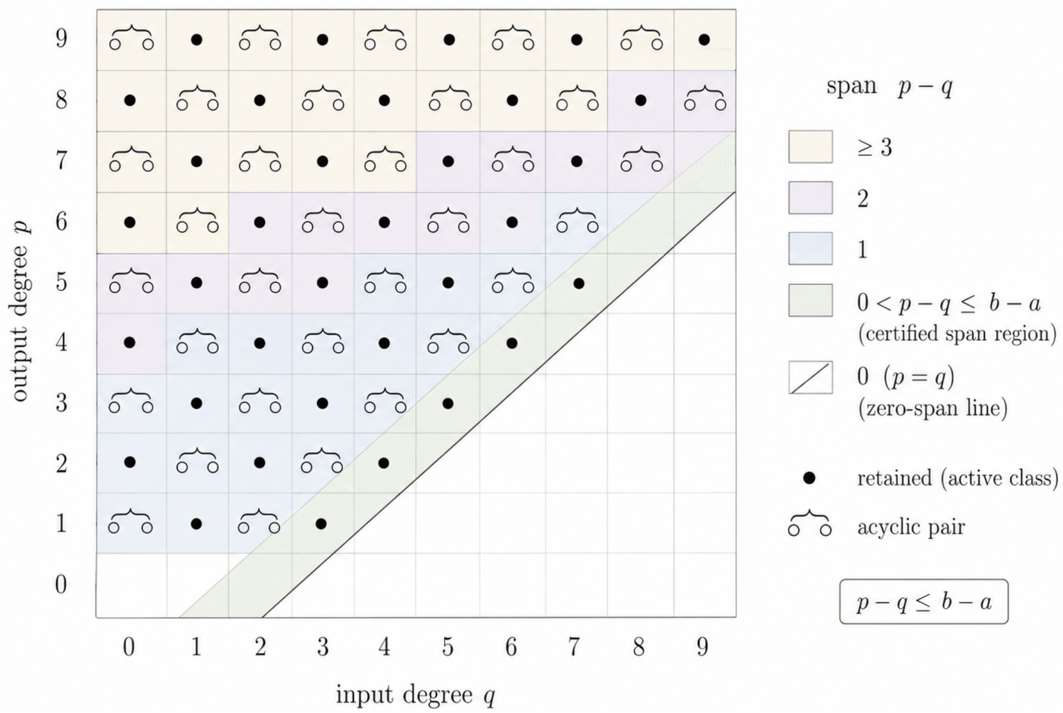

Figure 3: Span filtration.

Let b be the endpoint of the short rank window. Then the acyclic ideal is denoted by

$$\mathcal{I}^{\mathrm{ac}}_{a,b,k,t} \subset A^{\bullet}_{a,t}.$$

Here the ideal is generated by the Koszul-dual coordinate directions for the inactive acyclic pairs with span larger than $b-a$. Then the compressed algebra is

$$B^{\bullet}_{a,b,k,t} = P_{\leq k}\left(A^{\bullet}_{a,t}/\mathcal{I}^{\mathrm{ac}}_{a,b,k,t}\right),$$

where $P_{\leq k}$ keeps the part relevant for tangent cohomology up to degree k. Then the affine dg scheme $\widehat{\mathcal{M}}^{k}_{a,b,t} = \operatorname{Spec} B^{\bullet}_{a,b,k,t}$ is still finite since the uncompressed Hom-space algebra is finite over $[a,t]$. The ideal is equivariant since span and acyclicity depend on the intrinsic grading and differential, but not a chosen basis for N_s.

The coordinate-level impact is presented in Figure 4. The big block represents the Koszul-dual coordinate algebra; the small extracted piece represents the equivariant acyclic ideal; and the small block represents the kept bounded presentation. The classical admissible face remains present because the removed directions lie in negative dg degrees and are acyclic for the range under inspection.

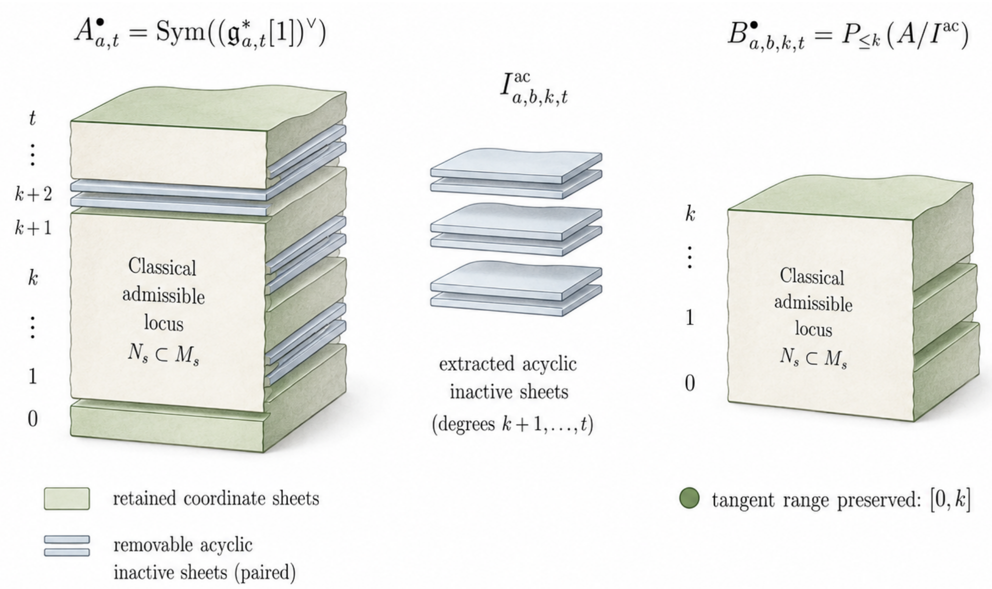

Figure 4: Acyclic compression.

This must be understood as a homological cancellation and not as a reduction of variables for convenience. A coordinate sheet will be eliminated only if the differential identifies it with a boundary counterpart. Surviving obstruction coordinates are retained even if their homogeneity span is large. This is the basic protection from loss of derived structure: compression comes after the differential, not after the algebraic picture size.

Lemma 3.1 (Span-acyclic cancellation)**.** *At a classical admissible point, the quotient map $A^{\bullet}_{a,t} \to B^{\bullet}_{a,b,k,t}$ induces an isomorphism on cotangent cohomology in degrees $[-k, 0]$ whenever all generators being eliminated belong to acyclic filtered pairs with associated classes which are not survived in this range.*

Proof. The associated graded terms of the filtered cotangent complex can be described via span spectral sequence. Elimination of a filtered acyclic pair is equivalent to the quotient of the complex by a contractible summand. Thus the latter quotient does not affect E_∞-terms in degrees $[-k, 0]$. Since the filtration is finite in the interval $[a, t]$, the convergence is finite and the induced map on cotangent cohomology is an isomorphism in the stated range. The duality argument applies to tangent cohomology in degrees $[0, k]$. □

4. Rank-sealed quotient over the short base

The compressed Maurer–Cartan algebra is not quite a derived Quot presentation yet. The admissible locus should encode that the chosen graded pieces are submodule pieces within M_*. Let

$$\phi^{\gamma}_{s} : N_s \longrightarrow M_s$$

be the map to the ambient graded piece, and let

$$\mu^{\gamma}_{s',s} : R_{s-s'} \otimes N_{s'} \longrightarrow N_s$$

be the multiplication map arising from the finite-stretch structure. The rank-sealing functional is

$$\rho_{a,b,t}(\gamma) = \sum_{s=b+1}^{t} \dim_{\mathbb{C}} \ker(\phi^{\gamma}_{s}) + \sum_{b<s'\leq s\leq t} \dim_{\mathbb{C}} \operatorname{coker}(\mu^{\gamma}_{s',s}).$$

The constraint $\rho_{a,b,t}(\gamma) = 0$ encodes injectivity beyond b of the inclusion maps and generation of the higher graded pieces using the shorter interval. This is an algebraic condition because each summand is a rank condition on a finite dimensional vector space.

The rank-seal certificate shown in Figure 5 presents both exactness conditions side-by-side. Injectivity and generation are both needed. Injectivity alone would only tell us that the pieces are contained in M_s; generation alone would allow collapse in the ambient module. The vanishing of both conditions seals the short-window construction by proving the propagated stretch to be a submodule of the desired Hilbert polynomial.

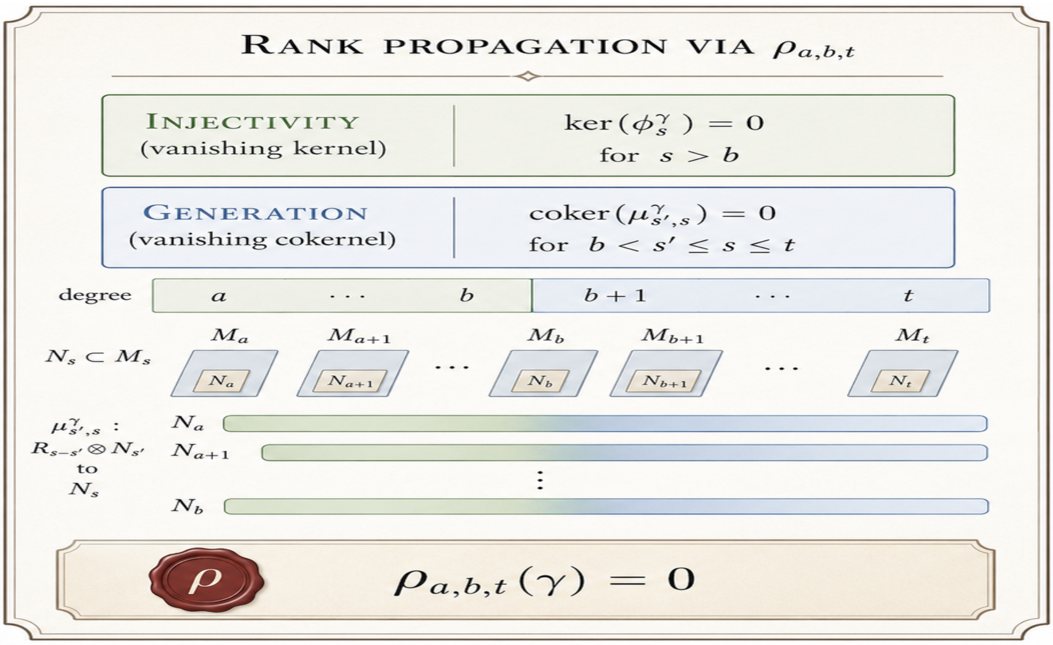

Figure 5: Rank certificate.

The certificate interpretation is precise, not heuristic. All kernel terms vanish implies that all higher maps $N_s \to M_s$ have maximal rank. All cokernel terms vanish implies that the multiplication maps cover the higher graded pieces. Therefore, the finite input over $[a,b]$ determines the admissible behavior over $[a,t]$ when the Maurer–Cartan equations and the rank functional coincide.

The quotient procedure consists of three steps. The partial quotient projects the injective embedding space to the product of Grassmannians by modding out the change-of-basis action. The algebraic quotient considers invariant functions after affine modification. Invariant theory in classical sense separates this affine construction from the geometric one [21]. Geometric invariant theory characterizes the stable loci where quotient fibers coincide with the group orbits [3]. This is critical since the dg structure sheaf fails to descend correctly in general outside the rank-certified open locus.

The three quotient steps are illustrated in Figure 6. The main object here is the rank-certified locus, while the adjacent vertices are associated with the partial, algebraic and geometric quotient steps. The triangular shape is significant; no quotient step is superior in the construction. One needs the correct quotient face at the right moment.

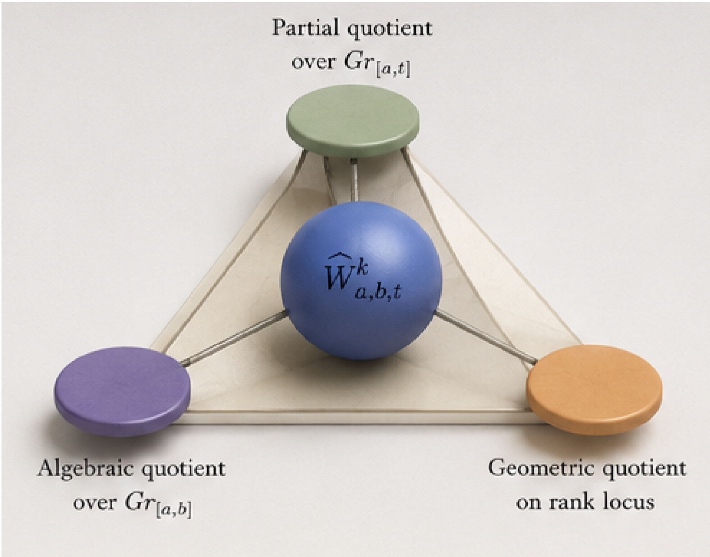

Figure 6: Quotient faces.

This construction explains the choice of $\mathrm{Gr}_{[a,b]}$ as the base for the compressed model. The former takes invariants prematurely missing some non-geometric fibers, the latter keeps the full product $\mathrm{Gr}_{[a,t]}$ failing to incorporate rank propagation and spectral acyclicity data. The rank-certified open locus is the place where both criteria hold: the algebra is sufficiently small to be affine over the short base and the quotient is geometric over the admissible locus.

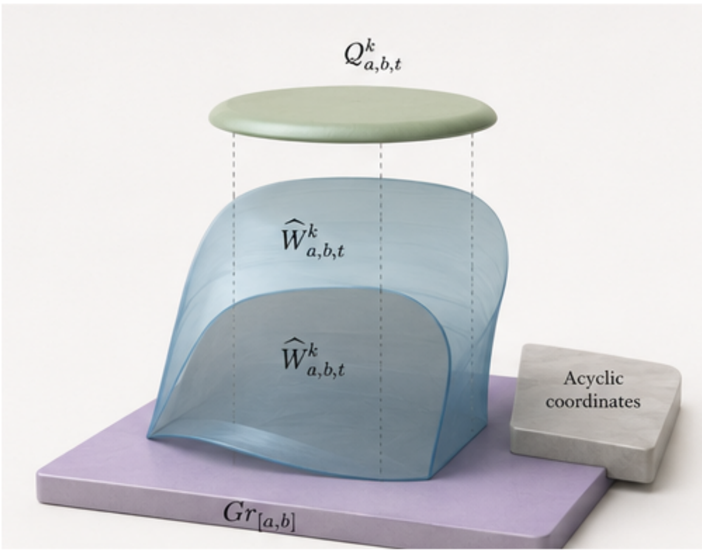

Figure 7: Compressed quotient.

The compressed quotient is the stack

$$\mathfrak{Q}^k_{a,b,t} = \widehat{W}^k_{a,b,t}/G_{a,b},$$

where $\widehat{W}^k_{a,b,t} \subset \widehat{\mathcal{M}}^k_{a,b,t}$ is the closed subset where $\rho_{a,b,t} = 0$. The changes in basis in higher degrees are captured by the affine coordinates and acyclic pairs, while the remaining group over the short window is quotiented over $\mathrm{Gr}_{[a,b]}$. This stack is affine over $\mathrm{Gr}_{[a,b]}$ as a dg stack and comes with the inherited differential from the compressed Maurer–Cartan algebra.

The geometry of $\mathfrak{Q}^k_{a,b,t}$ is illustrated in Figure 7. The base is the short Grassmannian, the middle object is the compressed affine dg fiber, and the top object is the quotient representation. The acyclic directions are placed outside the fiber since they no longer appear in the structure sheaf.

The figure is a concise summary of the construction. Since the lower base carries honest Grassmannian geometry, the construction stays rooted in classical Quot parameters. Since the fiber is compressed by acyclic cancellation alone, the bounded tangent theory remains in tact. The quotient lives over the short base because the rank certificate has already used the higher-degree admissibility to make exactness conditions apparent on $[a, b]$.

Table 3: Compression certificate.

Certificate component	Algebraic test	Consequence
Span acyclicity	Removed coordinates form filtered acyclic pairs outside $[-k, 0]$ cotangent degrees.	Tangent cohomology in $[0, k]$ is preserved.
Rank sealing	$\rho_{a,b,t}(\gamma) = 0$ on classical compressed solutions.	Higher pieces inject into M_s and are generated by multiplication.
Equivariance	$\mathcal{I}^{\mathrm{ac}}_{a,b,k,t}$ is stable under the residual basis-change action.	Compression is compatible with quotient formation.
Geometric orbit separation	Maximal-rank points have quotient fibers equal to group orbits.	Classical admissible quotient points are not over-identified.
Coherent generation	Cohomology sheaves over $\mathrm{Gr}_{[a,b]}$ are coherent in each retained degree.	A finite-type dg manifold replacement exists.

These certificate entries are the finite answer to the bounded presentation problem. Each row controls one possible source of failure: loss of tangent information, failure of rank propagation, incompatibility with group action, incorrect orbit separation, or lack of finite-type replacement. Only when all five tests are passed is a compressed quotient accepted. This allows the construction to have a concrete stopping condition in place of an aesthetic one.

5. Comparison results and interpreted outcomes

Theorem 5.1 (Bounded equivalence)**.** *Assume that the span-acyclic, rank-sealing, equivariant, quotient-separating, and coherent generation conditions in Table 3 hold for (a, b, t, k). Then the compressed quotient $\mathfrak{Q}^k_{a,b,t}$ and the uncompressed finite-stretch derived Quot presentation over $[a, t]$ have the same classical admissible quotient points and isomorphic tangent cohomology in degrees $[0, k]$ at corresponding points.*

Proof. The quotient by $\mathcal{I}^{\mathrm{ac}}_{a,b,k,t}$ removes only acyclic coordinate pairs. By the span-acyclic cancellation lemma, the cotangent cohomology in degrees $[-k, 0]$ and hence the tangent cohomology in degrees $[0, k]$ are unchanged. The rank functional then identifies the same classical admissible locus because $\rho_{a,b,t} = 0$ forces all higher maps $N_s \to M_s$ to be injective and all relevant multiplication maps to be surjective. Equivariance ensures that the compression commutes with the residual change-of-basis action. At points of maximum rank, the quotient is geometric, so group orbits are separated exactly as in the admissible finite-stretch Quot quotient. These statements combined yield the asserted equivalence of classical Quot points and the tangent cohomology isomorphism. □

This theorem is the main theorem of the paper. It is the finite criterion in its full form: a shorter interval determines the bounded derived Quot presentation as long as spectral acyclicity and rank sealing hold. The result is not that every long interval is superfluous. It is that the long interval contains only contractible coordinate directions and rank-generated affine coordinates after the certificate is closed.

The comparison of tangent spaces is illustrated in Figure 8. The two neighborhoods are the local models of the uncompressed and compressed presentations at a classical admissible point. The central acyclic kernel corresponds to the cotangent data removed by compression, and the aligned degree markers demonstrate that tangent cohomology up to degree k is preserved.

The local comparison is the principal deformation-theoretic outcome. Classical admissible points alone are not enough to distinguish the compressed dg model from a regular truncation of equations. Agreement of tangent complexes proves that the first-order deformations and obstruction layers through the chosen degree are preserved. This is why the construction remains derived even after compression: the negative degree part of the dg structure sheaf which matters to the chosen degree range survives.

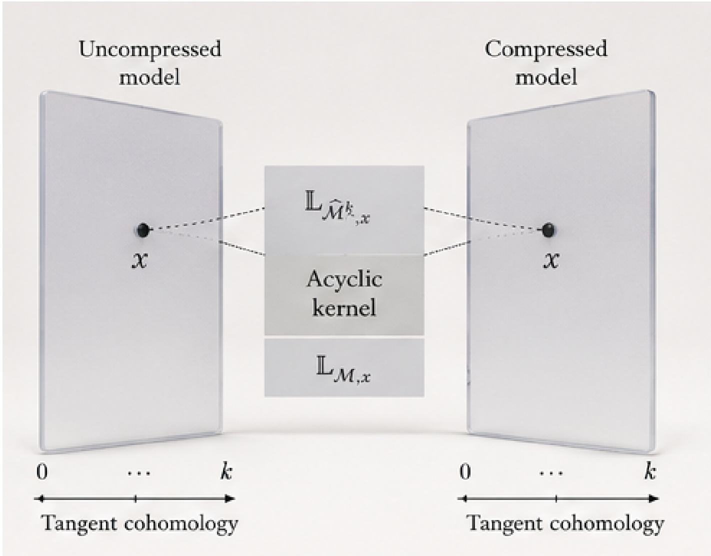

Figure 8: Local tangent comparison.

Proposition 5.1 (Finite-type replacement)**.** *If the cohomology sheaves of $\mathcal{O}^{\bullet}_{\mathfrak{Q}^k_{a,b,t}}$ are coherent over the smooth projective base $\mathrm{Gr}_{[a,b]}$ in the retained degrees, then $\mathfrak{Q}^k_{a,b,t}$ is weakly equivalent to a dg manifold of finite type.*

Proof. The morphism $\mathfrak{Q}^k_{a,b,t} \to \mathrm{Gr}_{[a,b]}$ is affine in the dg sense. The coherent cohomology sheaves over a projective base admit resolutions by locally free coherent sheaves. Adding such generators inductively in non-positive degrees yields an almost free differential graded structure sheaf whose cohomology sheaves coincide with those of $\mathcal{O}^{\bullet}_{\mathfrak{Q}^k_{a,b,t}}$. The resulting dg manifold has a smooth quasi-projective classical base and coherent locally free generators in every retained negative degree, hence it is a dg manifold of finite type. The construction yields a weak equivalence since it preserves the cohomology sheaves of the dg structure sheaf. $\square$

The finite-type replacement in Figure 9 replaces coherent cohomology sheaves by locally free dg generators. This step is not part of the span certificate audit but it converts the compressed dg scheme to a dg manifold of finite type needed for derived Quot representability. The base stays the same short Grassmannian interval.

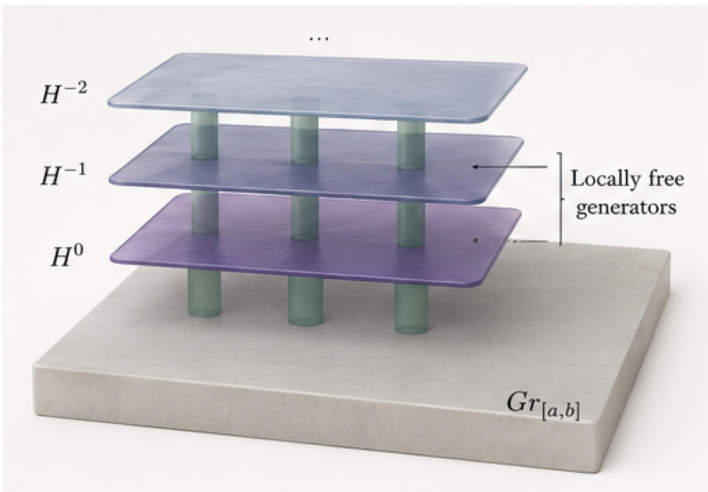

Figure 9: Finite-type replacement.

The sheaf scaffolding in the figure above provides an interpretation of finite type in concrete terms. Coherent cohomology is not left as a purely abstract sheaf property. It is replaced by locally free generators of an almost free dg algebra structure. This replacement is essential as the aim of the paper is to construct a differential graded manifold presentation, not a homotopy class of derived schemes.

The first mathematical outcome of the certificate audit is preservation of classical admissibility. The compressed quotient has no additional classical points since the rank sealing ensures that every point satisfies the injectivity and generation requirements of the finite-stretch Quot problem. It also has no missing admissible points since every uncompressed admissible solution is mapped to a rank-sealed compressed solution. This explains the role of $\rho_{a,b,t}$ — it is not an auxiliary tool but an exact finite condition linking the short base with the long stretch.

The second outcome is the preservation of bounded deformation theory. The removal of an acyclic ideal in a dg algebra can be either harmless or destructive, depending on the range of cotangent degrees involved. The span spectral sequence provides a way to make this distinction. If the removed pairs are invisible to the $[-k, 0]$ cotangent cohomology, then the tangent complex in degrees $[0, k]$ remains unchanged. This is a stronger statement than the equality of dimensions, since it describes the cohomology groups of the tangent complex, not just the alternating Euler characteristic.

The third outcome is the compatibility with the geometry of the Quot space. The construction does not simply take invariant functions on a quasi-affine object and hope for the proper moduli interpretation. The classical invariant theory warns that spectra of invariant functions may fail to separate individual orbits [21]. Geometric invariant theory remedies the problem on the stable or maximal-rank loci [3]. The construction thus works on this locus and uses the algebraic quotient only as an affine replacement consistent with the dg structure.

The fourth outcome is the finite-type representation in case of coherent generation. Once the compressed quotient is affine over $\mathrm{Gr}_{[a,b]}$ and its cohomology sheaves are coherent, locally free generators give a finite-type dg manifold replacement [22]. This replacement is used in the derived Quot representability theorem to give a finite dg presentation to the finite-stretch construction [23].

The certificate in Theorem form is presented in Figure 10. The central sector shows the bounded equivalence, and the other four sectors show the mathematical justifications for it: span acyclicity, rank sealing, quotient separating, and coherent generation. The outer band indicates the two preserved outputs: classical points and tangent cohomology up to degree k.

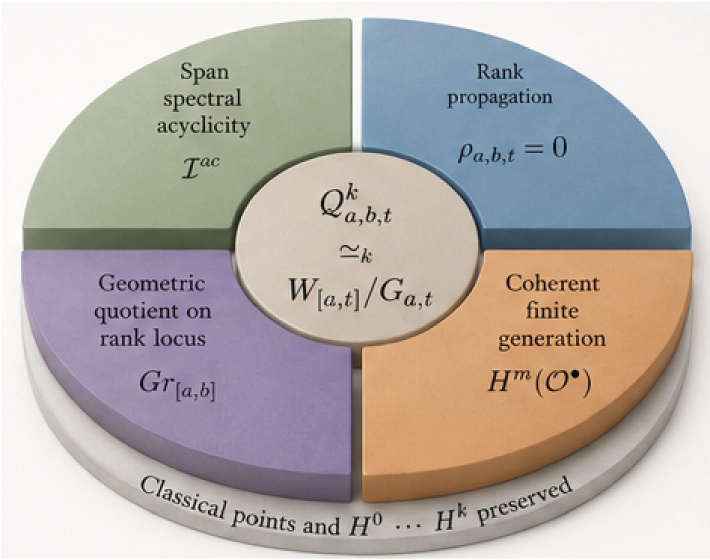

Figure 10: Equivalence certificate.

The certificate picture should be interpreted as the logical closure of the paper and not as a procedure diagram. Every sector corresponds to a necessity for the mathematics of the construction. Failure of the acyclicity sector means loss of tangent information. Failure of the rank sector means the short interval does not determine higher admissibility. Failure of the quotient sector means incorrect moduli interpretation of orbits. Failure of coherent generation means that a finite-type dg manifold cannot be constructed by the given argument.

A final consequence regards the finite algebraic corpus given in Table 2. The corpus is not just auxiliary notation; it is the entire finite input that generates the bounded presentation. The ambient modules M_s fix the target of the quotient problem, the Hilbert polynomial dimensions N_s fix the dimensions, the Hom space algebra fixes the higher equations, and the rank tests fix the classical admissibility condition. The theorem shows that no further long window Grassmannian coordinates are needed after the corpus passes the certificate test. Thus the theorem gives a concrete meaning to its result: the same bounded tangent theory is reconstructed from a smaller base since the longer interval was broken into acyclic directions and rank-generated affine directions.

The literature coordinate system in Table 1 also gives a way to understand the interpretation of the result. From the perspective of classical Quot theory, the result is a statement regarding the necessity of the portion of the Grassmannian interval that remains once regularity forces the problem into finite degree. From the perspective of deformation theory, the result is a statement about preserving the appropriate part of the cotangent complex. From the perspective of invariant theory, the result is a statement about delaying the construction of a quotient until rank conditions isolate the orbit locus. These interpretations are compatible since the construction starts from the same finite set of Hom spaces encoding all three structures. What is new is the use of the homogeneity span to assign the part of the dg coordinate algebra for each interpretation.

The comparison theorem consists of local and global parts. Locally, the theorem identifies the tangent cohomology at the corresponding admissible points. This is the part described in Figure 8; the kernel removed is contractible in the relevant cotangent degrees, and thus the local derived deformation problem is unchanged up to degree k. Globally, the theorem identifies the classical quotient points modulo the residual group action. The global statement relies on Figure 6 and Figure 7; local tangent identification is not enough to guarantee quotient orbit separation. Thus the result combines local derived equivalence with global quotient compatibility.

The finite-type proposition adds a second level of interpretation. The result could preserve the tangent cohomology and yet fail to give a finite-type dg manifold since its structure sheaf might have uncontrolled cohomology over the base. The coherent-generation condition prevents this. The cohomology sheaves can then be resolved by the generators, shown in Figure 9. The proposition hence connects the rank-sealed span audit to the language of strict dg-manifolds required for finite-type representability. It is especially significant since derived schemes and dg manifolds are not equivalent; the latter requires a more rigid almost-free presentation over a smooth classical base.

The result can also be viewed as a precision statement. For $k = 0$, the certificate preserves the classical tangent space and the admissible quotient locus. For larger k, the same construction keeps successively deeper obstruction layers. The successful certificate for the degree-k case does not mean a successful certificate in degree-$k+1$ case, because new spectral classes could be preserved at the next cotangent degree. This is expected; finite derived presentations depend on the required deformation depth. The construction makes this explicit by attaching the desired tangent degree directly to the acyclic ideal $\mathcal{I}^{\mathrm{ac}}_{a,b,k,t}$.

The rank functional gives the strongest classical content of the result. In general, one can preserve tangent complexes and still leave the classical locus hard to interpret in derived construction. In this case, the vanishing of $\rho_{a,b,t}$ forbids this interpretation ambiguity. It implies that each higher piece is not only embedded in the ambient module but also generated by the multiplication structure. The compressed object is hence not only a dg algebra with the correct homology but also a quotient presentation whose closed points have the intended interpretation as admissible graded submodule points. That is why the construction is based specifically on derived Quot geometry and not an arbitrary filtered dg Lie algebra.

6. Conceptual consequences and limitations of the construction

The construction alters the standard Quot presentation hierarchy. The typical starting point is the Grassmannian presentation with Grassmannian intervals, equations, comparison of quotients, and stabilization. Rank-sealed span audit takes off from the Hom-space presentation and asks what operations are preserved under the span spectral sequence of interest. The order matters. Filtering comes before quotienting, hence identifying contractible long-span coordinates that will be filtered out of the long window even if they are passed through the quotients since they affect no derived structures in the bounded window.

The change of order here is not just a mere reordering of ingredients of the usual presentation. The Grassmannian presentation has a degree-zero preference, which stems from starting from the subspaces $N_s \subset M_s$. The Hom-space presentation starts with the maps which have to satisfy the Maurer-Cartan equation. Rank-sealed span audit uses the latter to refine the former by performing the filtering on the operation algebra and then returning to the Grassmannian base. This back-and-forth procedure is possible due to the fact that the construction remains faithful to the classical Quot points while shrinking the long-window dg coordinate algebra.

The approach is aligned with the deformation-theoretic tradition of controlling the moduli via dg Lie algebras, but the use of the homogeneity of the coordinate ring and the module M_* allows one to construct a filtration that cannot be found in a generic deformation functor. In a generic dg Lie algebra, one could impose various filtrations for various reasons. Here the filtration is imposed by the nature of the object being filtered. The span of the operation has a geometric interpretation as the measurement of the number of multiplication by R_+ needed to transfer information between graded pieces. The latter is the reason why the span spectral sequence can be coupled to the rank functional: a long-span operation is irrelevant only when it is acyclic according to the differential and admissible according to the rank condition.

Another aspect where the approach differs from the deformation functor-based techniques is the distinction between homogeneity length and deformation depth. The former measures the distance between the input degree and the output degree of the multilinear operation. The latter is the cohomological degree in the tangent complex. Although there is a relation between the two via the differential, they are not the same. The span audit exploits this inequality: a long-span operation can remain if it survives in the tangent range and a short-span operation can be removed if it is in the acyclic pair. The result is a more subtle finite-window criterion than just extending t until the stability is achieved.

The figures and the tables together demonstrate that the construction has the fixed mathematical corpus and the fixed certificate. The finite corpus (Table 2) specifies the exact objects being manipulated. The certificate (Table 3) identifies the exact tests that need to be performed. The graphics then distinguish these tests geometrically: the homogeneity window shows where the graded pieces lie, the operation ledger shows which Hom spaces contain equations, the span filtration shows what is the span of operations that need to be audited, the rank certificate shows when the admissibility propagates, and the quotient figures show when the moduli interpretation is correct. The integrated interpretation is important since no component is sufficient by itself.

The literary context also helps in specifying the scope of the result. Classical Quot theory provides projective parameter spaces but no tangent structure. The derived deformation theory provides tangent complexes but does not always provide the finite Grassmannian quotients. Invariant theory provides the quotients but requires careful consideration of geometric orbit separation. The finite-type dg manifolds supply the strict dg representatives but depend on coherent generation. The finite-stretch theory of Ciocan-Fontanine and Kapranov supplies the Hom-space algebra being used in the audit [16]. The finite-type representability theorem of Borisov, Katzarkov and Sheshmani supplies the finite-type dg-manifold end point [23]. The rank-sealed span audit connects these two stages in a single finite algebraic corpus.

There are some limitations. The certificate may fail for small b. In this case, the long-span spectral sequence contains active classes or the rank functional does not vanish on the compressed locus, and the short base does not define the bounded derived presentation. The construction does not state that every derived Quot problem has a finite representation. Instead it provides the exact condition for the existence of the finite presentation. This conditional character is mathematically useful in that it turns the search for the finite model into the testable finite certificate.

The second limitation concerns the quotient singularities outside of the maximal-rank locus. Algebraic quotients may exist as spectra of invariant functions, but fibers can contain several orbits. The construction avoids this by locating the geometric quotient on the rank-certified open locus. If the problem possesses the unstable behavior intersecting the admissible region, the further invariant-theoretic study is required. This is not a flaw of the span audit; rather it reflects the common difference between good and geometric quotients.

The third limitation concerns the selected tangent range. The equivalence is stated up to degree k. The increase in k may require increasing the short window $[a, b]$, the long interval $[a, t]$ or decreasing the acyclic ideal. The method is thus naturally compatible with the tower of precision levels. The certificate for degree $k + 1$ restricts to one for degree k if the same rank and quotient conditions are satisfied, but the converse need not to be true. Hence the construction is bounded by design, and its power consists in the ability to make this bound explicit.

The discussion also explains why the final conclusion is not the absolute statement about compression. The paper proves the precise algebraic assertion: the bounded derived Quot presentation over the long finite stretch is obtained from the shorter Grassmannian base if the long-span operations are spectrally inactive and the higher rank conditions propagate. The assertion holds under the certificate stated in Table 3, and no claim is made without this certificate. This is the proper level of generality for the finite-window construction in derived algebraic geometry.

7. Conclusion: finite-window validity

The main assertion is that the bounded derived Quot presentation over the long homogeneity interval $[a, t]$ is defined by the shorter Grassmannian interval $[a, b]$ once the Hom-space algebra and the rank conditions pass the rank-sealed span certificate. Spectral acyclicity guarantees that the long-span coordinates being removed do not affect the cotangent cohomology in degrees $[-k, 0]$. Rank sealing guarantees that the higher graded pieces inject into the ambient module and are generated by multiplication from the short interval. Quotient separation guarantees that the residual basis-change action has the proper orbit interpretation on the maximal-rank locus. Coherent generation guarantees that the resulting dg scheme admits the finite-type dg manifold representative.

This conclusion is specific to the construction of the paper. It does not just state that the finite approximations are useful, but rather that the finite, symbolic and checkable certificate can guarantee exactly that the short window already contains the bounded derived Quot information. The classical admissible locus and the tangent cohomology up to degree k are preserved because only contractible in the relevant cotangent range directions are discarded. Therefore the compressed quotient $\mathfrak{Q}^k_{a,b,t}$ is the valid finite differential graded presentation of the selected derived Quot structure under the algebraic conditions stated in Table 3.

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