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Approximation Method for Finding Fixed Point of Generalized Suzuki Nonexpansive Mappings on Hyperbolic Spaces

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Abstract

In this article, we aim to prove strong and Δ-convergence theorems of Noor iterative process for generalized Suzuki nonexpansive mappings (GSNM) on uniform convex hyperbolic spaces. Due to the richness of uniform convex hyperbolic spaces, the results of this paper can be used as an extension and generalization of many famous results in Banach spaces together with $CAT(0)$ spaces.

Keywords: Mappings, convergence, hyperbolic spaces, iteration process.

1. Introduction

In 2008, Suzuki in [1] introduced a family of single valued mappings as:

Definition 1.1. *Let us consider a Banach space $\mathbb{B}$ and a mapping $\mathfrak{F}$ on the subset $\mathbb{S}$ of $\mathbb{B}$ satisfying following condition:*

$$\frac{1}{2}\|u-\mathfrak{F}v\| \leq \|u-v\| \implies \|\mathfrak{F}u-\mathfrak{F}v\| \leq \|u-v\|, \tag{1}$$

$\forall\ u, v \in \mathbb{S}$.

The mapping $\mathfrak{F}$ work as intermediate class of mapping between non-expansiveness and quasi-non-expansiveness as given below:

Definition 1.2. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a Banach Space. Then $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ is non-expansive if $\|\mathfrak{F}u-\mathfrak{F}v\| \leq \|u-v\|\ \forall\ u, v \in \mathbb{S}$.*

Definition 1.3. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a Banach Space. Then $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ is quasi-nonexpansive if $\|\mathfrak{F}x-\rho\| \leq \|x-\rho\|$ for every $\rho \in FP(\mathfrak{F})$ and $\forall\ x \in \mathbb{S}$, moreover $FP(\mathfrak{F})$ represents fixed point set of $\mathfrak{F}$.*

Example 1.1. *Let $\mathfrak{F}$ on [0, 5] is defined by;*

$$\mathfrak{F}x = \begin{cases} 0, & x \neq 5; \\ 1, & x = 5. \end{cases}$$

Then clearly $\mathfrak{F}$ is not non-expansive but it satisfies condition (1).

Example 1.2. *Let $\mathfrak{F}$ on [0, 5] defined by*

$$\mathfrak{F}x = \begin{cases} 0, & x \neq 5; \\ 2, & x = 5. \end{cases}$$

Then $\mathfrak{F}$ fails to fullfill condition (1), however $\mathfrak{F}$ is quasi-non-expansive and $FP(\mathfrak{F}) = \{0\} \neq \phi$.

Suzuki [1] done significant work in showing the presence of the fixed point and convergence theorem in Banach spaces equipped with mapping satisfying condition (1).

In [2] Dhompongsa et al. enhanced the conclusions of Suzuki [1]with different conditions on Banach spaces and obtained a fixed point result in these spaces equipped with mapping satisfying condition C.

Nanjaras et al. [5] rendered sundry characterization of existing fixed point results equipped with mappings satisfying condition C in the skeleton of CAT(0) spaces. There is need to generalize the result of Suzuki type nonexpansive mappings which was efficiently done by Karapınar et al. [2] in 2011 as given bellow.

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Definition 1.4. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space, equipped with mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ and if*

$$\frac{1}{2}\rho(\mathfrak{F}u, \mathfrak{F}v) \leq \rho(u, v) \Rightarrow \rho(\mathfrak{F}u, \mathfrak{F}v) \leq \theta(u, v),$$

where $\theta(u, v) = max\{\rho(u, v), \rho(u, \mathfrak{F}u), \rho(v, \mathfrak{F}v), \rho(u, \mathfrak{F}v), d(v, \mathfrak{F}u)\}\ \forall\ u, v \in \mathbb{S}$. Then $\mathfrak{F}$ is considered to be a Suzuki-Ciric mapping (SCC) [3].

Definition 1.5. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space, equipped with mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ and if*

$$\frac{1}{2}\rho(\mathfrak{F}u, \mathfrak{F}v) \leq \rho(u, v) \Rightarrow \rho(\mathfrak{F}u, \mathfrak{F}v) \leq \nu(u, v),$$

where $\nu(u, v) = \max\left\{\rho(u, v), \frac{\rho(u,\mathfrak{F}u)+\rho(v,\mathfrak{F}v)}{2}, \frac{\rho(u,\mathfrak{F}v)+\rho(v,\mathfrak{F}u)}{2}\right\}\ \forall\ u, v \in \mathbb{S}$.

Then $\mathfrak{F}$ is considered to be a Suzuki-KC mapping (SKC).

Definition 1.6. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space, equipped with mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ and if*

$$\frac{1}{2}\rho(\mathfrak{F}u, \mathfrak{F}v) \leq \rho(u, v) \Rightarrow \rho(\mathfrak{F}u, \mathfrak{F}v) \leq \frac{\rho(u, \mathfrak{F}u) + \rho(v, \mathfrak{F}v)}{2},$$

$\forall\ u, v \in \mathbb{S}$.

Then $\mathfrak{F}$ is considered to be a Kannan-Suzuki mapping (KSC).

Definition 1.7. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space, equipped with mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ and if*

$$\frac{1}{2}\rho(\mathfrak{F}u, \mathfrak{F}v) \leq \rho(u, v) \Rightarrow \rho(\mathfrak{F}u, \mathfrak{F}v) \leq \frac{\rho(v, \mathfrak{F}u) + \rho(u, \mathfrak{F}v)}{2},$$

$\forall\ x, y \in \mathbb{S}$.

Then $\mathfrak{F}$ is considered to be a Chatterjea-Suzuki mapping (CSC).

Clearly every nonexpansive mapping is SKC, but the converse may not true [3].

Example 1.3. *Set $\mathfrak{F}$ on [0, 6] by:*

$$\mathfrak{F}x = \begin{cases} 0, & x \neq 6; \\ 1, & x = 6. \end{cases}$$

Clearly $\mathfrak{F}$ is not non-expansive but $\mathfrak{F}$ fullfill both the SCC and SKC conditions.

Example 1.4. *Set R on [0, 6] by:*

$$Sx = \begin{cases} 0, & x \neq 6; \\ 3, & x = 6. \end{cases}$$

Clearly R does not fulfill the SKC condition, moreover R is quasi-nonexpansive and $FP(R) \neq \phi$.

Example 1.5. *Let the space $\mathbb{B} = (0, 0), (0, 1), (1, 1), (1, 2)$ with l^∞ metric:*

$$\rho((u_1, v_1), (u_2, v_2)) = \max\{|u_1 - v_1|, |v_1 - v_2|\}.$$

Set $\mathfrak{F}$ on $\mathbb{B}$ by:

$$\mathfrak{F}x = \begin{cases} (1, 1), & if (u, v) \neq (0, 0); \\ (0, 1), & if (u, v) = (0, 0). \end{cases}$$

Clearly $\mathfrak{F}$ fullfill SKC's condition. Assume that $(u, v) = (0, 0)$ and $(u, v) = (1, 1)$, then

$$\frac{1}{2}\rho(\mathfrak{F}(0, 0), (0, 0)) \leq \rho((0, 0), (1, 1))$$

and

$$\begin{aligned} \nu((0, 0), (1, 1)) = \max\{&\rho((0, 0), (1, 1)), \frac{1}{2}[\rho(\mathfrak{F}(0, 0), (0, 0)), \rho(\mathfrak{F}(1, 1), (1, 1))] \\ &, \frac{1}{2}[\rho(\mathfrak{F}(1, 1), (0, 0)), \rho(\mathfrak{F}(0, 0), (1, 1))]\} \\ &= 1, \end{aligned}$$

thus

$$\rho(\mathfrak{F}(0, 0), \mathfrak{F}(1, 1)) = 1 \leq \nu((0, 0), (1, 1)) = 1.$$

Clearly SKC condition is fulfilled by other points in $\mathbb{B}$.

Moreover $FP(\mathfrak{F}) = (1, 1) \neq \phi$, and $FP(\mathfrak{F})$ is convex and closed.

This is significant to understand the different iterative process adapted by several writers in locating fixed points of the space equipped with nonlinear mappings, moreover solution of their operator equations.

The iteration process introduced by Mann (see [6, 7]) is explained below:

Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is convex and $\mathbb{B}$ is Banach Space, and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be a nonlinear mapping, for every point $u_0 \in \mathbb{S}$, the sequence u_n in $\mathbb{S}$ is manufactured by

$$u_{n+1} = (1 - \gamma_n)u_n + \gamma_n \mathfrak{F} u_n = M(u_n, \gamma_n, \mathfrak{F}), n \in N,$$

is called Mann Iterative process.

It should be noted that γ_n represents a real sequence in [0, 1] which fulfill the conditions given below:
(M_1): $0 \leq \gamma_n < 1$,
(M_2): $\lim_{n\to\infty} \gamma_n = 0$,
(M_3): $\sum_{n=1}^{\infty} \gamma_n = \infty$.
one can replace M_3 by $\sum_{n=1}^{\infty} \gamma_n(1 - \gamma_n) = \infty$ in other applications.

The Ishikawa introduced an iteration process which improves the Mann iteration process (see [6, 8]) as follows:

Setting $\mathbb{S}, \mathbb{B}$, and $\mathfrak{F}$ as in (M), for every point $u_0 \in \mathbb{S}$, the sequence u_n in $\mathbb{S}$ is manufactured by:

$$u_{n+1} = (1 - \gamma_n)u_n + \gamma_n \mathfrak{F}((1 - \alpha_n)u_n + \alpha_n \mathfrak{F} u_n), \;\; n \in N,$$

is called Ishikawa iterative process, where γ_n and α_n are sequences in [0, 1] which satisfy the following conditions:
(I_1): $0 \leq \gamma_n \leq \alpha_n < 1$,
(I_2): $\lim_{n\to\infty} \alpha_n = 0$,
(I_3): $\sum_{n=1}^{\infty} \gamma_n \alpha_n = \infty$.

Some authors switch condition $(I_1) 0 \leq \gamma_n \leq \alpha_n < 1$, with the general condition $(I_1^{'}) 0 < \gamma_n, \alpha_n < 1$, and notice that, with this switching, the iterative process manufactured by Ishikawa (I) is a spontaneous generalization of the iterative process manufactured by Mann (M). It is perceived that, if the iterative process manufactured by Mann (M) is convergent, then the iterative process manufactured by Ishikawa (I) through condition $(I_1^{'})$ is also convergent, with appropriate conditions on γ_n and α_n.

Recently, Agarwal et al. [14] broached the S-iteration process which is independent of above two iterative process as follows:

For $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is convex and $\mathbb{B}$ is linear space, and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be a mapping, for every point $u_0 \in \mathbb{S}$, the iterative sequence u_n in $\mathbb{S}$ is manufactured by S-iteration process is given below:

$$\begin{cases} u_{n+1} = (1 - \gamma_n)u_n + \gamma_n \mathfrak{F} u_n \\ v_n = (1 - \alpha_n)u_n + \alpha_n \mathfrak{F} u_n, \qquad n \in N, \end{cases}$$

where γ_n and α_n are sequences in (0, 1) filling the condition:

$$\sum_{n=0}^{\infty} \gamma_n \alpha_n (1 - \alpha_n) = \infty.$$

It is perceived that both the S-iteration process and the Picard has same rate of convergence, which is rapid than the iteration process manufactured by Mann equipped with contraction mapping (see [6, 14, 15]).

We use the definition of a hyperbolic space given in [17], [18] and [19], because the definition given by Reich and Shafrir [24] is a bit more repressive. The hyperbolic spaces in the Reich and Shafrir sense [24] is unbounded by taking family of metric lines M instead of metric segments. Every subset of hyperbolic space is hyperbolic itself by the definition which we consider and it gives convergence result too.

Definition 1.8. *Consider the metric space* $(\mathbb{B}, \rho)$ *equipped with convex mapping* $\Omega : \mathbb{B}^2 \times [0, 1]$ *then the triplet* $(\mathbb{B}, \rho, \Omega)$ *is said to be hyperbolic space if it fulfills the conditions given below:*
$(\Omega_1) : \rho(x, \Omega(u, v, \gamma)) \leq \gamma\rho(x, u) + (1 - \gamma)d(x, v);$
$(\Omega_2) : \rho(\Omega(u, v, \gamma), \Omega(u, v, \alpha)) = \gamma - \alpha|\rho(u, v);$
$(\Omega_3) : \Omega(u, v, \gamma) = \Omega(v, u, 1 - \gamma);$
$(\Omega_4) : \rho(\Omega(u, w, \gamma), \Omega(v, y, \gamma)) \leq (1 - \gamma)\rho(u, v) + \gamma\rho(w, y),$
$\forall$ *u, v, x and $y \in \mathbb{B}$ and $\gamma, \alpha \in [0, 1]$.*

Takahashi manufacture the convex metric space [20], in which the triplet $(\mathbb{B}, \rho, \Omega)$ *fulfills* Ω_1. *Goebel and Kirk in [22] manufacture their own definition of above space, where triplet* $(\mathbb{B}, \rho, \Omega)$ *filling conditions* (Ω_1)-(Ω_3).

Reich and Shafrir [24] and Kirk [25] manufactured their definition of hyperbolic space by using 'condition III' of Itoh [23] which is equivalent to Ω_4.

The class of hyperbolic spaces are rich in nature and contains different spaces, manifold of the Hadamard type and convex subsets thereof, for more see [26], and the CAT(0) spaces along with Ω *as the unique geodesic path between any two points in* $\mathbb{B}$*. Bruhat and Tits [11] shows that hyperboloic space is a* $CAT(0)$*-space if and only if it fulfills the so called CN-inequality.*

Wataru Takahashi [19] introduce the notion of convex set $\mathbb{S}$ *of hyperbolic spaces* $\mathbb{B}$ *if it satisfies the following condition* $\Omega(u, v, \gamma) \in \mathbb{S}\ \forall\ u, v \in \mathbb{S}$ *and* $\gamma \in [0, 1]$*. We often use the notion* $(1 - \mu)u \oplus \mu v$ *for* $\Omega(u, v, \mu)$*,*$\forall\ u, v \in \mathbb{B}$ *and* $\mu \in [0, 1]$*.*

Assume $\forall\ u, v \in \mathbb{B}$ *and* $\mu \in [0, 1]$*. Setting*

$$\rho(u, (1 - \mu)u \oplus \mu v) = \mu\rho(u, v)$$

and

$$\rho(v, (1 - \mu)u \oplus \mu v) = (1 - \mu)\rho(u, v)$$

which is considered to be more general setting of a convex metric space [20, 21].

A hyperbolic space $(\mathbb{B}, \rho, \Omega)$ *is uniformly convex in the sense of [16] if, for any* $q > 0$ *and* $\epsilon \in (0, 2]$*, there exists* $\delta \in (0, 1]$ *such that,* $\forall\ c, u, v \in \mathbb{B}$*,*

$$\rho(\frac{1}{2}u \oplus \frac{1}{2}v, b) \leq (1 - \delta)q,$$

provided $\rho(u, c) \leq q, \rho(v, c) \leq r,$ *and* $\rho(u, v) \geq \epsilon q$*.*

Setting $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ *equipped with* $\delta = \eta(q, \varepsilon)$ *such that* $q > 0$ *and* $\varepsilon \in (0, 2]$ *then* η *is said to be modulus of uniform convexity. Clearly with this setting if* q *decreases for stationary* ε *then* η *is monotone.*

The aim of this article is to prove strong convergence and $\Delta-$*convergence of Noor iterative process for GSNM in uniform convex hyperbolic spaces. Due to the richness of these spaces our results can be used as an extension and generalization of famous results in Banach and* $CAT(0)$ *spaces (see [2-7,16,22,29-31]).*

2. Preliminaries

First, we recall the notion of Δ-convergence and few of its primary characteristics.

Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space and let u_n be any sequence in Moreover, diam($\mathbb{S}$) signify the diameter of $\mathbb{S}$. Set a continuous functional $r_b(., u_n) : \mathbb{B} \to \mathbb{R}^+$ as

$$r_b(u, \{u_n\}) = \lim_{n\to\infty} \sup \rho(u_n, u), u \in \mathbb{B}.$$

The asymptotic radius of u_n is signified by $r_b(\mathbb{S}, \{u_n\})$ in connection with $\mathbb{S}$ and is defined to be the infimum of $r_b(., u_n)$ over $\mathbb{S}$.

Furthermore, if

$$r_b(w, \{u_n\}) = \inf\{(u, \{u_n\}) : u \in \mathbb{S}\},$$

then the point $w \in \mathbb{S}$ signify as an asymptotic center of the sequence u_n in connection with $\mathbb{S}$.

AC($\mathbb{S}, u_n$) signifies the set of all asymptotic centers of u_n in connection with $\mathbb{S}$, which is the set of minimizers of the functional $r(, \{u_n\})$ and it can be empty or a singleton or contain infinitely many points.

The notions $r_b(\mathbb{B}, u_n) = r_b(\{u_n\})$ and AC($\mathbb{B}, \{u_n\}$) = AC($\{u_n\}$), respectively, signifies asymptotic radius and asymptotic center taken in connection with $\mathbb{B}$.

Clearly, for $u \in \mathbb{B}, r_b(u, \{u_n\}) = 0$ if and only if $\lim_{n\to\infty} u_n = u$.

Moreover every sequence which is bounded has a unique asymptotic center in connection with each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces.

The following lemma is due to Leuştean [31] and we know that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.1. *[31] Assume* $\mathbb{S} \subset \mathbb{B}$*, where* $\mathbb{S}$ *is nonempty , moreover* $\mathbb{S}$ *is also closed and convex. Furthermore, the triplet* $(\mathbb{B}, \rho, \Omega)$ *represents uniformly convex hyperbolic space, which is complete and having* η *as monotone modulus of uniform convexity. Then every sequence* $\{u_n\}$ *in* $\mathbb{B}$*, which is bounded has a unique asymptotic center referring* $\mathbb{S}$ *as defined above.*

Definition 2.1. *Let* $\mathbb{B}$ *be hyperbolic space and* u_n *is any sequence in* $\mathbb{B}$*. If* u *is the unique asymptotic center of every subsequence* t_n *of* u_n *then* $\{u_n\}$ *is considered to be* $\Delta -$ *convergent to* $u \in \mathbb{B}$*. In such a case, we set* $\Delta - lim_n u_n = u$ *and we refer* u *the* $\Delta - limit$ *of* u_n*.*

Lemma 2.2 (33)**.** *The triplet* $(\mathbb{B}, \rho, \Omega)$ *represents uniformly convex hyperbolic space having* η *as monotone modulus of uniform convexity. Moreover assume* $u \in \mathbb{B}$ *and* $\{s_n\}$ *be a sequence in* $[c, d]$ *with* $0 < c, d < 1$*. If* $\{u_n\}$ *and* $\{v_n\}$ *are any two sequences in* $\mathbb{B}$ *so that*
$\limsup_{n\to\infty} \rho(u_n, u) \leq e,$
$\limsup_{n\to\infty} \rho(v_n, p) \leq e,$
$\lim_{n\to\infty} \rho(\Omega(u_n, v_n, s_n), u) = e,$
for some $e \geq 0$*, then* $\lim_{n\to\infty} \rho(u_n, v_n) = 0$*.*

3. Main results

Now we will give the definition of $Fe\acute{j}er$ monotone sequences.

Definition 3.1. *Assume* $\mathbb{S} \subset \mathbb{B}$ *, where* $\mathbb{S}$ *is nonempty and* $\mathbb{B}$ *is a hyperbolic space. Moreover , suppose that* $\{u_n\}$ *be a sequence in* $\mathbb{B}$*. Then the sequence* $\{u_n\}$ *is said to be* $Fe\acute{j}er$ *monotone in connection with* $\mathbb{S}$ *if* $\forall\ u \in \mathbb{S}$ *and* $n \in N$*,*

$$\rho(u_{n+1}, u) \leq \rho(u_n, u).$$

Proposition 3.1. *[19] Assume* $\mathbb{S} \subset \mathbb{B}$ *, where* $\mathbb{S}$ *is nonempty and* $\mathbb{B}$ *is a hyperbolic space. Moreover , suppose that* u_n *be a* $Fe\acute{j}er$ *monotone sequence in connection with* $\mathbb{S}$*. Then the following conditions hold:*
(1) $\{u_n\}$ *is bounded;*
(2) the sequence $\{\rho(u_n, t)\}$ *is decreasing and convergent* $\forall\ t \in FP(\mathfrak{F})$*.*

We are now able to present the iterative process manufactured by Noor in hyperbolic spaces (see [19]):

Assume $\mathbb{S} \subset \mathbb{B}$*, where* $\mathbb{S}$ *is nonempty, moreover* $\mathbb{S}$ *is close and convex, and* $\mathbb{B}$ *is hyperbolic space. Furthermore,* $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ *be a mapping. For any* $u_1 \in \mathbb{S}$*, the sequence* u_n *of the Noor iteration process is manufactured by:*

$$\begin{cases} u_{n+1} = \Omega(u_n, \mathfrak{F}v_n, \gamma_n), \\ v_n = \Omega(u_n, \mathfrak{F}w_n, \alpha_n), \\ w_n = \Omega(u_n, \mathfrak{F}u_n, \beta_n), \qquad n \in N, \end{cases} \tag{2}$$

where γ_n *and* α_n *are real sequences such that* $0 < a \leq \gamma_n, \alpha_n$ *and* $\beta_n \leq b < 1$*.*

We are able to manufacture the proof of the following lemma from the definition of SKC-mapping.

Lemma 3.1. *Assume* $\mathbb{S} \subset \mathbb{B}$*, where* $\mathbb{S}$ *is nonempty and* $\mathbb{B}$ *is a hyperbolic space. Moreover suppose that* $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ *be an SKC-mapping. If* u_n *be sequence manufactured by* (3.1)*, then* u_n *is* $Fe\acute{j}er$ *monotone sequence in connection with* $FP(\mathfrak{F})$*.*

Proof. Let $q \in FP(\mathfrak{F})$. Then by (3.1), we have

$$\begin{aligned} \rho(w_n, q) &= \rho(\Omega(u_n, \mathfrak{F}u_n, \beta_n), q) \\ &\leq (1 - \beta_n \rho(u_n, q) + \beta_n \rho(\mathfrak{F}u_n, q) \\ &\leq (1 - \beta_n \rho(u_n, q) + \beta_n [5\rho(q, \mathfrak{F}_q) + (u_n, q)] \\ &\leq (1 - \beta_n \rho(u_n, q) + \beta_n \rho(u_n, q) \\ &\leq \rho(u_n, q). \end{aligned} \tag{3}$$

$$\begin{aligned} \rho(v_n, q) &= \rho(\Omega(u_n, \mathfrak{F}w_n, \alpha_n), q) \\ &\leq (1 - \alpha_n \rho(u_n, q) + \alpha_n \rho(\mathfrak{F}w_n, q) \\ &\leq (1 - \alpha_n \rho(u_n, q) + \alpha_n [5\rho(q, \mathfrak{F}_q) + (w_n, q)] \\ &\leq (1 - \alpha_n \rho(u_n, q) + \alpha_n \rho(w_n, q) \\ &\leq (1 - \alpha_n \rho(u_n, q) + \alpha_n \rho(u_n, q) \\ &\leq \rho(u_n, q). \end{aligned} \tag{4}$$

$$\begin{aligned} \rho(u_{n+1}, q) &= \rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), q) \\ &\leq (1 - \gamma_n \rho(u_n, q) + \gamma_n \rho(\mathfrak{F}v_n, q) \\ &\leq (1 - \gamma_n \rho(u_n, q) + \gamma_n [5\rho(q, \mathfrak{F}_q) + (v_n, q)] \\ &\leq (1 - \gamma_n \rho(u_n, q) + \gamma_n \rho(v_n, q) \\ &\leq (1 - \gamma_n \rho(u_n, q) + \gamma_n \rho(u_n, q) \end{aligned}$$

$$\leq \rho(u_n, q). \tag{5}$$

$\forall\ q \in FP(\mathfrak{F})$. Which completes the proof. $\square$

Lemma 3.2. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty , moreover $\mathbb{S}$ is also close and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space , which is complete and having η as monotone modulus of uniform convexity and set $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC-mapping. If the sequence u_n is manufactured by (3.1), then $FP(\mathfrak{F})$ is nonempty if and only if u_n is bounded and $\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0$.*

Proof. Suppose $FP(\mathfrak{F})$ be nonempty and $q \in FP(\mathfrak{F})$. Then, the sequence u_n is $Fe\acute{j}er$ monotone with respect to $FP(\mathfrak{F})$ by using by Lemma 3.3. Furthermore, u_n is bounded and $\lim_{n\to\infty} \rho(u_n, q)$ by using Proposition 3.2.

Set $\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = e \geq 0$. If $e = 0$, then clearly we have

$$\begin{aligned} \rho(u_n, \mathfrak{F}u_n) &\leq \rho(u_n, q) + \rho(\mathfrak{F}u_n, q) \\ &\leq \rho(u_n, q) + 5\rho(q, \mathfrak{F}_q) + \rho(u_n, q) \\ &\leq 2\rho(u_n, q) \end{aligned}$$

Applying the limit supremum, we have

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0.$$

Set $e > 0$. Moreover $\mathfrak{F}$ is an SKC-mapping, then

$$\rho(\mathfrak{F}_q, \mathfrak{F}v_n) \leq \rho(q, v_n)$$

and

$$\rho(\mathfrak{F}_q, \mathfrak{F}u_n) \leq \rho(q, u_n).$$

Therefore,

$$\begin{aligned} \rho(\mathfrak{F}_n, q) &\leq \rho(\mathfrak{F}_n, \mathfrak{F}q). \\ &\leq \rho(u_n, q) \end{aligned}$$

for every $n \in N$. Applying the limit supremum, we get

$$\limsup_{n\to\infty} \rho(\mathfrak{F}_n, q) \leq e,$$

for $e > 0$. Further we have

$$\limsup_{n\to\infty} \rho(\mathfrak{F}_v, q) \leq e.$$

Applying the limit supremum, we get

$$\limsup_{n\to\infty} \rho(v_n, q) \leq e.$$

Since

$$\begin{aligned} e &= \limsup_{n\to\infty} \rho(u_{n+1}, q) \\ &\leq \limsup_{n\to\infty} \{\rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), q)\} \\ &\leq \limsup_{n\to\infty} \{(1-\gamma_n)\rho(u_n, q) + \gamma_n \rho(\mathfrak{F}v_n, q)\} \\ &\leq (1-\gamma_n) \limsup_{n\to\infty} \rho(u_n, q) + \gamma_n \limsup_{n\to\infty} \rho(\mathfrak{F}v_n, q) \end{aligned}$$

we have

$$e \leq ((1-\alpha_n)e + \alpha_n e) = e.$$

Thus

$$\lim_{n\to\infty} \{\rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), q)\} = e,$$

for $e > 0$. Consequently it occurs from the lemma(2.3) that

$$\lim_{n\to\infty} \rho(\mathfrak{F}u_n, \mathfrak{F}v_n) = 0.$$

Next,

$$\begin{aligned}\rho(u_{n+1}, \mathfrak{F}u_n) &= \rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), \mathfrak{F}u_n)\\ &\leq d\rho(\mathfrak{F}v_n, \mathfrak{F}u_n)\\ &\to 0 \qquad\qquad as : n \to \infty.\end{aligned}$$

Hence, we have

$$\begin{aligned}\rho(u_{n+1}, \mathfrak{F}v_n) &= \rho(u_{n+1}, \mathfrak{F}u_n) + (\mathfrak{F}u_n), \mathfrak{F}v_n)\\ &\to 0 \qquad\qquad as : n \to \infty.\end{aligned}$$

Notice that

$$\begin{aligned}\rho(u_{n+1}, q) &= \rho(u_{n+1}, \mathfrak{F}v_n) + (\mathfrak{F}v_n), q)\\ &\leq \rho(u_{n+1}, \mathfrak{F}v_n) + (\mathfrak{F}_n), q)\end{aligned}$$

Which produces

$$c \leq \liminf_{n\to\infty} \rho(v_n, q).$$

From above inequalities, we get

$$\lim_{n\to\infty} \rho(v_n, q) = e.$$

Thus, we get

$$\lim_{n\to\infty} \{\rho(\Omega(u_n, \mathfrak{F}u_n, \alpha_n), q)\} = e,$$

which implies

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0.$$

Conversely, assume that the sequence $\{u_n\}$ is bounded and $\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0$.
Set $AC(\mathbb{S}, \{u_n\}) = u$ be a singleton. Then $u \in \mathbb{S}$. Further $\mathfrak{F}$ is an $SKC - mapping$

$$d(u_n, \mathfrak{F}u) \leq 5\rho(u_n, \mathfrak{F}u_n) + \rho(u_n, u),$$

which implies that

$$\begin{aligned}r_b(\mathfrak{F}u, u_n) &= \limsup_{n\to\infty} \rho(u_n, \mathfrak{F}u)\\ &\leq \limsup_{n\to\infty} [5\rho(u_n, \mathfrak{F}u_n) + \rho(u_n, u)]\\ &\leq \limsup_{n\to\infty} \rho(u_n, u)\\ &= r_b(u, u_n).\end{aligned}$$

By utilizing the uniqueness of the asymptotic center, $\mathfrak{F}u = u$, so u is a fixed point of $\mathfrak{F}$. Consequently, $FP(\mathfrak{F})$ is nonempty. □

Now, we are able to prove the $\Delta - convergence$ theorem.

Theorem 3.1. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty , moreover $\mathbb{S}$ is also close and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space , which is complete and having η as monotone modulus of uniform convexity and set $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an $FP(\mathfrak{F}) \neq \phi$. If the sequence u_n is manufactured by (3.1), then the sequence u_n is $\Delta - convergent$ to a fixed point of $\mathfrak{F}$.*

Proof. Suppose $\mathfrak{F}$ is an SKC-mapping. We observe that u_n be a bounded sequence. Therefore, u_n has a $\Delta - convergent$ subsequence.We have to show that every $\Delta - convergent$ subsequence of u_n has a unique $\Delta - limit$ in $FP(\mathfrak{F})$. To prove this claim, suppose s and t be $\Delta - limits$ of the subsequences s_n and t_n of u_n, respectively. Since $AC(\mathbb{S}, s_n) = s$ and $AC(\mathbb{S}, t_n) = t$ by using lemma 2.1. Now by lemma 3.3, s_n is a bounded sequence and and $\lim_{n\to\infty} \rho(s_n, \mathfrak{F}s_n) = 0$.
We have to show that s is a fixed point of $\mathfrak{F}$.

$$d(s_n, \mathfrak{F}s) \leq 5\rho(s_n, \mathfrak{F}s_n) + \rho(s_n, s).$$

Applying the limit supremum, we get

$$r_b(s_n, \mathfrak{F}s) = \limsup_{n\to\infty} \rho(s_n, \mathfrak{F}s)$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty}[5\rho(s_n, \mathfrak{F}s_n) + \rho(s_n, s)] \\
&\leq \limsup_{n\to\infty} \rho(s_n, s) \\
&= r_b(s_n, s).
\end{aligned}
$$

Hence, we have

$$r_b(s_n, \mathfrak{F}s) \leq r_b(s_n, s).$$

By uniqueness of the asymptotic center, $\mathfrak{F}s = s$.

By using same argument, we can show that $\mathfrak{F}t = t$. Consequently, s and t are fixed points of $\mathfrak{F}$. Now we show that $s = t$. Suppose on contrary that $s \neq t$, moreover by the uniqueness of the asymptotic center,

$$
\begin{aligned}
\limsup_{n\to\infty} \rho(u_n, s) &= \limsup_{n\to\infty} \rho(s_n, s) \\
&< \limsup_{n\to\infty} \rho(s_n, t) \\
&= \limsup_{n\to\infty} \rho(u_n, t) \\
&= \limsup_{n\to\infty} \rho(t_n, t) \\
&< \limsup_{n\to\infty} \rho(t_n, s) \\
&= \limsup_{n\to\infty} \rho(u_n, s)
\end{aligned}
$$

which is a contradiction. Therefore s = t. □

Now, we will introduce the strong convergence theorems in hyperbolic spaces.

Theorem 3.2. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty , moreover $\mathbb{S}$ is also close and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space , which is complete and having η as monotone modulus of uniform convexity and set $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC-mapping.If the sequence u_n is manufactured by (3.1), then the sequence u_n converges strongly to some fixed point of $\mathfrak{F}$ if and only if*

$$\liminf_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0,$$

where $D(u_n, FP(\mathfrak{F})) = \inf_{u\in FP(\mathfrak{F})} \rho(u_n, u)$.

Proof. Clearly the necessity condition is trivial. The prove completes only by showing the sufficient condition. To manufacture the proof first, we show that $FP(\mathfrak{F})$ is closed. Assume that $\mathfrak{F}$ is SKC-mapping, moreover suppose that u_n be any sequence in $FP(\mathfrak{F})$ which converges to some point $u \in \mathbb{S}$.

$$\rho(u_n, \mathfrak{F}u) \leq 5\rho(\mathfrak{F}u_n, \mathfrak{F}u) + \rho(u_n, u) \leq \rho(u_n, u).$$

Applying the limit, we get

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u) \leq \lim_{n\to\infty} \rho(u_n, u) = 0.$$

Since, the limit is unique, so we get $u = \mathfrak{F}u$, which shows that $FP(\mathfrak{F})$ is closed.

Assume that

$$\liminf_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0,$$

Moreover, we obtain

$$D(u_{n+1}, FP(\mathfrak{F})) \leq D(u_n, FP(\mathfrak{F}))$$

Thus $\lim_{n\to\infty} \rho(u_n, FP(\mathfrak{F}))$ exists by applying Lemma 3.3 and using Proposition 3.2. Consequently we know that

$$\lim_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0.$$

Consequently, we can set a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ so that

$$\rho(u_{n_k}, q_k) < \frac{1}{2^k},$$

for every $k \geq 1$, where $\{q_k\} \in FP(\mathfrak{F})$.

Applying Lemma 3.3, we get

$$\rho(u_{n_{k+1}}, q_k) \leq \rho(u_{n_k}, q_k) < \frac{1}{2^k}$$

from which we can deduce that

$$\begin{aligned}\rho(q_{k+1}, q_k) &\leq \rho(q_{k+1}, u_{n_{k+1}}) + \rho(u_{n_{k+1}}, q_k)\\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k}\\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Thus $\{q_k\}$ is a Cauchy sequence. Whereas $FP(\mathfrak{F})$ is closed. Then $\{q_k\}$ is a convergent sequence.

Suppose $\lim_{k\to\infty} q_k = q$. Then the prove completes by showing that $\{u_n\}$ converges to q. In fact, whereas

$$\rho(u_{n_k}, q) \leq \rho(u_{n_k}, q_k) + \rho(q_k, q) \to 0$$

as $k \to \infty$.

We have

$$\lim_{k\to\infty} \rho(u_{n_k}, q) = 0.$$

Since $\lim_{k\to\infty} \rho(u_n, q)$ exists, the sequence $\{u_n\}$ is converges to q. □

Next, we will give one more strong convergence theorem by using Theorem 3.8. We call up the definition of condition (I) broached by Senter and Doston [34].

Assume $(\mathbb{B}, \rho)$ be a metric space and $\mathbb{S} \subset \mathbb{B}$ which is nonempty, equipped with a mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$. Then $\mathfrak{F}$ is claimed to fulfill condition (I), if $\exists$ a nondecreasing function $f[0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0 \; \forall \; t \in (0, \infty)$ so that

$$\rho(u, \mathfrak{F}u) \geq f(D(u, FP(\mathfrak{F}))),$$

$\forall \; u \in \mathbb{S}$, where $D(u, FP(\mathfrak{F})) = \inf d(u, q) : q \in FP(\mathfrak{F})$.

Theorem 3.3. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty , moreover $\mathbb{S}$ is also close and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space , which is complete and having η as monotone modulus of uniform convexity and set $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC-mapping with condition (I) and$FP(\mathfrak{F}) \neq \phi$. Then the sequence $\{u_n\}$ which is manufactured by (3.1) converges strongly to some fixed point of $\mathfrak{F}$.*

Proof. From Theorem 3.6, and applying Lemma 3.4, we have

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0.$$

The condition (I) gives us

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) \geq \lim_{n\to\infty} f(D(u_n, FP(\mathfrak{F}))),$$

for $f[0, \infty) \to [0, \infty)$,which is nondecreasing with $f(0) = 0$, $f(t) > 0$ for t, such that $0 < t < \infty$.

Consequently, we get

$$\lim_{n\to\infty} f(D(u_n, FP(\mathfrak{F}))) = 0.$$

Whereas f is a nondecreasing mapping filling $f(0) = 0$ for every t, such that $0 < t < \infty$, we get

$$\lim_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0.$$

Which completes the proof from Theorem. □

4. Numerical example

Example 4.1. *Consider the real line R with usual metric ρ defined as $\rho(u, v) = |u - v|$, moreover suppose $\mathbb{S} = [0, 4] \subset R$. Set*

$$\Omega(u, v, \gamma) = \gamma u + (1{-}\gamma v),$$

for every $u, v \in \mathbb{S}$.

Then (R, ρ, Ω) is a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity and clearly $\mathbb{S} \subset R$, which is nonempty close and convex. Set a mapping $\mathfrak{F}$ as manufactured in Example 1.6.

Clearly $\mathfrak{F}$ fulfills the SKC condition with $0 \in \mathbb{S}$ as a fixed point of $\mathfrak{F}$. Moreover , it is noticed that it fulfills all conditions in Theorem 3.6. Suppose γ_n and α_n be constant sequences such that $\gamma_n = \alpha_n = \beta_n = \frac{1}{2}$ for every $n \geq 0$. We encounter following cases for $\mathfrak{F}$.

Case 1: Set u $\neq 4$, for the sake of simplicity, we suppose that $u_0 = 1$. Moreover by the iterative process manufactured in (3.1) and the definition of Ω, we get

$$\begin{aligned} w_0 &= \Omega(u_0, \mathfrak{F}u_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}u_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}u_0) \end{aligned}$$

and

$$\begin{aligned} v_0 &= \Omega(u_0, \mathfrak{F}w_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}w_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}w_0) \end{aligned}$$

and

$$\begin{aligned} u_1 &= \Omega(u_0, \mathfrak{F}v_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}v_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}v_0) \end{aligned}$$

Case 2: Set u $= 4$, for the sake of simplicity, we suppose that $u_0 = 4$. Moreover by the iterative process manufactured in (3.1) and the definition of Ω, we get

$$\begin{aligned} w_0 &= \Omega(u_0, \mathfrak{F}u_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}u_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}u_0) \end{aligned}$$

and

$$\begin{aligned} v_0 &= \Omega(u_0, \mathfrak{F}w_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}w_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}w_0) \end{aligned}$$

and

$$\begin{aligned} u_1 &= \Omega(u_0, \mathfrak{F}v_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}v_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}v_0) \end{aligned}$$

$w_1 = \Omega(u_1, \mathfrak{F}u_1, \frac{1}{2})$
$v_1 = \Omega(u_1, \mathfrak{F}w_1, \frac{1}{2})$
$u_2 = \Omega(u_1, \mathfrak{F}v_1, \frac{1}{2})$

Consequently by simple calculations, it can be seen that the sequence $\{u_n\}$ converges to $0 \in FP(\mathfrak{F})$.

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