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Entropy of Weighted Graph by Edge Weights by Using Sombor Reduced Index

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Abstract

This paper delves into the exploration of entropy in weighted graphs, where edge weights are determined by the Sombor Reduced Index. The primary aim is to establish and substantiate results by examining various graph types, including Connected Graphs, Star Graphs, Unicyclic Graphs, Regular Graphs, Complete Graphs, Complete Bipartite Graphs, Chemical Graphs, and Tree Graphs. Through rigorous analysis, we unveil the implications of these weighted structures on information content and structural complexities. Additionally, we extend our study to compute weighted entropies for molecular graphs representing specific dendrimer structures. This research contributes to a nuanced understanding of the informational intricacies embedded in diverse graph configurations, particularly emphasizing their significance in the molecular domain.

Keywords: Star Graph, Unicyclic Graph, Regular Graph, Complete Graph, Complete Bipartite Graph.

1. Introduction

A Graph Γ is an arranged pair of sets $\zeta(\Gamma)$ and $\xi(\Gamma)$. The components of $\zeta(\Gamma)$ are named as vertices and the component of $\xi(\Gamma)$ are named as edges. A line joining two dots (vertices) is called an edge and is denoted by $\xi(\Gamma)$. Simply we denote an Edge as rs instead of $\{r, s\}$. In a graph dots represent the vertex and we denote a vertex set by $\zeta(\Gamma)$. All vertices adjacent(means associated with same edges) to dots(vertex) r are neighbors of s. The neighborhood of r is the arrangement of the neighbors of r. The vertex having 0 degree is called a isolated vertex. The total vertex in a graph tell us the order of a graph and is represented by $\mid \Gamma \mid=\mid \zeta \mid$. The total edges in a graph gives the size of a graph and is denoted by $\parallel \Gamma \parallel=\parallel \xi \parallel$. An edge having starting and ending point is same is called loop. Those edges having same pair of vertices called multiple edges. The quantity of edges associated with dot(vertex) ζ_i (implies the number of edges associated with vertex) and is signified as $\deg \zeta_i$. Since each line is connected with two vertices. A dot(vertex) having maximum lines (edges) tell us the maximum degree and is denoted by $\theta(G)$ and defined as $\theta(G) = \max\{\deg r \mid r \in \zeta(\Gamma)\}$. The Minimum degree of a graph is the minimum degree(minimum number of lines connected) of its vertex, is denoted by $\phi(\Gamma)$ and is defined as $\phi(\Gamma) = \min\{\deg r \mid r \in \zeta(\Gamma)\}$. A basic graph on 'p' vertices in which every two vertices are connected by an edge is called Complete graph and is denoted by K_p. In K_p all vertices have same degree.

$$\begin{aligned}
&\parallel K_p \parallel = {}^pC_2 = \frac{p(p-1)}{2}\\
&\quad and\\
&\mid K_p \mid = p\\
&\quad \phi(\Gamma) = \theta(\Gamma) = \deg(G) = p-1
\end{aligned}$$

A graph in which every vertex are connected with equal number of edges (means that have same degree)is called Regular graph. If every vertex has m-edges(or degree m), then these types of graphs are ordinary graph of degree m or m-regular. If the vertex set of Graph Γ can be divided into two sets(having no common vertex) A and B so that each line(edge) of Γ joins a vertex of A with a vertex of B, then Γ is a Bipartite Graph and is indicated by $K_{q,p}$. A Bipartite Graph in which every vertex in A is joined to every vertex in B with the help of one edge and represented by $K_{q,p}$. A graph Γ is connected if the any two vertex are connected by a path or a line(edge). Or in other words, A graph is connected if each pair of vertex is joined by a way. A unicyclic graph is an connected graph containing precisely one cycle. A connected unincyclic graph is in this manner a pseudo tree that isn't a tree. Unicyclic graph is indicated by $U_{p,m}$. A tree is an undirected(which contains no direction) graph in which any two vertices are connected by precisely one way. A tree graph is indicated by T_p. A star graph is the $K_{1,p}$ of 'p' vertices and is meant by S_p. Some creators characterize S_p to be tree of order p. A graph related

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with compounds in which molecules are considers as vertices and bonds are Considers as edges. The topological index is a genuine symbol related with the sub-atomic graph. Many topological indices are characterized in [3]- [5]. Few of them depend on distance, but others depend on degree and have discovered numerous applications in drug store.

In 1975, the primary degree based topological index [6] was present.Latter, this record was summed up anyone genuine number α by Estrada *et al* in [7] and is said to Randić index.Another well known Topological index based on the vertex degree of the Graph is the sombor index [1]. In light of the historic work of Shannon's [14], in the last part of the 1950s started to examine the entropy estimation of organization system. Rashevsky utilizes the idea of graph entropy to quantify the primary intricacy of chart on the based of Shannon's entropy. Mowshowitz [15] also present the entropy as data hypothesis.Mowshowitz [16] later contemplated the numerical properties of graph entropy and applications. Another are in Körners entropy [17]. Many kind of graph conclusion have been used to expand graph entropy, for example, eigenvalue and network data [18], based on distance of graph entropy and based on degree of graph entropy.

We utilize the idea of graph entropy as a weighted graph, just as Dehmer [18]. Few of them based on degree are described by examining limits of entropy of certain class of graphs [20], [21]. In [22] Chen *et al.* also present the idea of graph entropy. As of late, Gutman presented another topological index under the name of Sombor reduced index $SO_{rd}(\Gamma)$.

In this paper we study the graph entropy by taking Sombor reduced index with edge weights and prove some external properties of graph entropy for special families of graphs such as connected graph, regular graph, complete bipartite graph, chemical graph, tree graph, unicyclic graph and star graphs.

2. Preliminaries

Definition 2.1. *Sombor Reduced Index Let $\Gamma = (\zeta, \xi)$ is a limited simple graph and is characterized as [23]*

$$SO_{rd}(\Gamma) = \sum_{rs \in \xi(\Gamma)} \sqrt{(d_r - 1)^2 + (d_s - 1)^2}$$

where d_r denote the degree of the vertex r in $SO_{rd}(\Gamma)$.

Definition 2.2. *Entropy*

The entropy of edge weighted Graph $\Gamma = (\zeta, \xi, w_t)$ is characterized by

$$E(\Gamma, w_t) = -\sum_{rs \in \xi(\Gamma)} X_{r,s} \log X_{r,s}$$

where,

$$X_{r,s} = \frac{w_t(rs)}{\sum_{rs \in \xi(\Gamma)} w_t(rs)}$$

Definition 2.3. *Weighted Graph A graph in which a weight gives to each edge.*

Definition 2.4. *Entropy Of Weighted Graphs Graph have been started in the late of 1950s dependent on the fundamental work because of Shannon. For model, diagram entropy measures have been widely to portray the design of diagram based frameworks in numerical, science and in software engineering related areas [1]. Rashevsky is the primary who presented the purported underlying data content dependent on parcel on vertex orbits [10]. Mowshowitz utilized a few measures and demonstrated a few properties [9]. For a given graph Γ and vertex s_i, let d_i be the degree of s_i.For an edge $s_i s_j$, one defines.*

$$X_{ij} = \frac{w_t(s_i s_j)}{\sum_{j=1}^{d_i} w_t(s_i s_j)}$$

where, $w_t(s_i s_j)$ is the weight of edge $s_i s_j$ and $w_t(s_i s_j) > 0$. The vertex entropy defined as

$$N(s_i) = \sum_{j=1}^{d_i} X_{ij} \log(X_{ij})$$

3. Main Results

Theorem 3.1. *If a connected graph Γ consist on p-vertices for $p \geq 3$. Then*

$$\log SO_{rd}(\Gamma) - \log \sqrt{2}.(p-2) \leq E(\Gamma, SO_{rd}(\Gamma)) \leq \log(SO_{rd}(\Gamma))$$

Proof. If Γ is a simple connected graph consist on p-vertices, then the maximum degree is $(p-1)$ and minimum degree is 1 with any edge rs, the minimum possible degree of r and s are 1 and 2 respectively, and the maximum possible degrees of r and s are $(p-1)$ and $(p-1)$ so we have,

$$SO_{rd}(\Gamma) = \sum_{rs\epsilon\xi} \sqrt{(d_r-1)^2+(d_s-1)^2}$$

$$E(\Gamma, SO_{rd}(\Gamma)) = -\sum_{rs\epsilon\xi} \frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}} \log \frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}}$$

$$E(\Gamma, SO_{rd}(\Gamma)) = -\sum_{rs\epsilon\xi} \frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}}$$

$$\left(\log\sqrt{(d_r-1)^2+(d_s-1)^2} - \log\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}\right)$$

$$= -\sum_{rs\epsilon\xi} \frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}} \log\sqrt{(d_r-1)^2+(d_s-1)^2} +$$

$$\sum_{rs\epsilon\xi} \frac{\sqrt{d_r^2+d_s^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}} \log\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}$$

$$= \log\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2} - \log\sqrt{(d_r-1)^2+(d_s-1)^2}$$

$$E(\Gamma, SO_{rd}(\Gamma)) = \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2}$$

SO,

$$\begin{aligned}
E(\Gamma, SO_{rd}(\Gamma)) &\le \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\
&= \log SO_{rd}(\Gamma) - \frac{1}{2}\log((2-1)^2+(1-1)^2) \\
&= \log SO_{rd}(\Gamma) - \log\sqrt{1} \\
&= \log SO_{rd}(\Gamma) - \log 1 \\
&= \log SO_{rd}(\Gamma) \\
E(\Gamma, SO_{rd}(\Gamma)) &\le \log SO_{rd}(\Gamma)
\end{aligned}$$

Also,

$$\begin{aligned}
E(\Gamma, SO_{rd}(\Gamma)) &\ge \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\
&= \log SO_{rd}(\Gamma) - \frac{1}{2}\log((p-1-1)^2+(p-1-1)^2) \\
&= \log SO_{rd}(\Gamma) - \frac{1}{2}\log 2(p-2)^2 \\
&= \log SO_{rd}(\Gamma) - \log\sqrt{2(p-2)^2} \\
&= \log SO_{rd}(\Gamma) - \log\sqrt{2}.(p-2) \\
E(\Gamma, SO_{rd}(\Gamma)) &\ge \log SO_{rd}(\Gamma) - \log\sqrt{2}.(p-2)
\end{aligned}$$

Hence,

$$\log SO_{rd}(\Gamma) - \log\sqrt{2}.(p-2) \le E(\Gamma, SO_{rd}(\Gamma)) \le \log(SO_{rd}(\Gamma))$$

□

Theorem 3.2. *Suppose Γ is a regular graph having p-vertices. Let ϕ and θ be the minimum and maximum degree of Γ, respectivly. Then,*

$$\log(SO_{rd}(\Gamma)) - \log(\phi-1)\sqrt{2} \le E(\Gamma, SO_{rd}(\Gamma)) \le \log SO_{rd}(\Gamma) - \frac{1}{2}\log(\theta-1)\sqrt{2}$$

Proof. A regular graph having p-vertices, then it has degree m and has $r = \frac{1}{2}p(m-1)$ edges then its sombor reduced index is

$$SO_{rd}(\Gamma) = \frac{1}{2}p(m-1)\sqrt{(m-1)^2+(m-1)^2}$$

$$\frac{p(\phi-1)^2}{\sqrt{2}} \leq SO_{rd}(\Gamma) \leq \frac{p(\theta-1)^2}{\sqrt{2}}$$

$$\begin{aligned} E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\ &= \log SO_{rd}(\Gamma) - \frac{1}{2}\log((\theta-1)^2+(\theta-1)^2) \\ &= \log SO_{rd}(\Gamma) - \frac{1}{2}\log 2(\theta-1)^2 \\ E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log(\theta-1)\sqrt{2} \end{aligned}$$

Also,

$$\begin{aligned} E(\Gamma, SO_{rd}(\Gamma)) &\geq \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\ &= \log SO_{rd}(\Gamma) - \frac{1}{2}\log((\phi-1)^2+(\phi-1)^2) \\ &= \log SO_{rd}(\Gamma) - \frac{1}{2}\log 2(\phi-1)^2 \\ E(\Gamma, SO_{rd}(\Gamma)) &\geq \log SO_{rd}(\Gamma) - \log(\phi-1)\sqrt{2} \end{aligned}$$

Hence,

$$\log(SO_{rd}(\Gamma)) - \log(\phi-1)\sqrt{2} \leq E(\Gamma, SO_{rd}(\Gamma)) \leq \log SO_{rd}(\Gamma) - \frac{1}{2}\log(\theta-1)\sqrt{2}$$

□

Theorem 3.3. *For a regular graph $\Gamma = (\zeta, \xi, w_t)$ having p-vertices such that $p \geq 3$,*

$$\log p \leq E(\Gamma, SO_{rd}(\Gamma)) \leq \log\frac{p(p-1)}{2}$$

and $\log p = E(\Gamma, SO_{rd}(\Gamma)) \Leftrightarrow \Gamma$ *is cyclic and* $E(\Gamma, SO_{rd}(\Gamma)) = \log\frac{p(p-1)}{2} \Leftrightarrow \Gamma$ *is complete.*

Proof. If Γ is a m-Regular Graph with $m \geq 2$.As Γ is connected with $p \geq 3$, so

$$\begin{aligned} E(\Gamma, SO_{rd}(\Gamma)) &= -\sum_{rs\epsilon\xi}\frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}}\log\frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}} \\ E(\Gamma, SO_{rd}(\Gamma)) &= -\sum_{rs\epsilon\xi}\frac{\sqrt{(m-1)^2+(m-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(m-1)^2+(m-1)^2}}\log\frac{\sqrt{(m-1)^2+(m-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(m-1)^2+(m-1)^2}} \\ &= -\frac{p(m-1)}{2}.\frac{\sqrt{(m-1)^2+(m-1)^2}}{\frac{p(m-1)}{2}.\sqrt{(m-1)^2+(m-1)^2}}\log\frac{\sqrt{(m-1)^2+(m-1)^2}}{\frac{p(m-1)}{2}\sqrt{(m-1)^2+(m-1)^2}} \\ &= -\log\frac{1}{\frac{p(m-1)}{2}} \\ &= -\log\frac{2}{p(m-1)} \\ &= \log\frac{p(m-1)}{2} \end{aligned}$$

$E(\Gamma, SO_{rd}(\Gamma)) \leq \log\frac{p(p-1)}{2} \Leftrightarrow m = p$ and $E(\Gamma, SO_{rd}(\Gamma)) \geq \log p \Leftrightarrow m = 3$. Hence,

$$\log p \leq E(\Gamma, SO_{rd}(\Gamma)) \leq \log\frac{p(p-1)}{2}$$

□

Theorem 3.4. *In a complete bipartite graph Γ with p-vertices. Then*

$$\log(p-1) \leq E(\Gamma, SO_{rd}(\Gamma)) \leq \log\left(\left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil\right)$$

and $\log(p-1) = E(\Gamma, SO_{rd}(\Gamma)) \Leftrightarrow \Gamma$ *is star graph and* $E(\Gamma, SO_{rd}(\Gamma)) = \log\left(\left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil\right) \Leftrightarrow \Gamma$ *is complete bipartite graph.*

Proof. In a complete bipartite graph consist on p-vertices and having two parts m and n vertices, respectively. Therefore, $(m-1)+(n-1)=p$

$$\begin{aligned}
E(\Gamma, SO_{rd}(\Gamma)) &= -\sum_{rs\epsilon\xi} \frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}} \log \frac{\sqrt{(d_r-1)^2+(d_s-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(d_r-1)^2+(d_s-1)^2}} \\
E(\Gamma, SO_{rd}(\Gamma)) &= -\sum_{rs\epsilon\xi} \frac{\sqrt{(m-1)^2+(n-1)^2}}{\sum_{rs\epsilon E}\sqrt{(m-1)^2+(n-1)^2}} \log \frac{\sqrt{(m-1)^2+(n-1)^2}}{\sum_{rs\epsilon\xi}\sqrt{(m-1)^2+(n-1)^2}} \\
&= -(m-1)(n-1).\frac{\sqrt{(m-1)^2+(n-1)^2}}{(m-1)(n-1).\sqrt{(m-1)^2+(n-1)^2}}. \\
&\log \frac{\sqrt{(m-1)^2+(n-1)^2}}{(m-1)(n-1)\sqrt{(m-1)^2+(n-1)^2}} \\
&= -\log \frac{1}{(m-1)(n-1)} \\
&= \log{(m-1)(n-1)}
\end{aligned}$$

Moreover, $\log(p-1) = E(\Gamma, SO_{rd}(\Gamma)) \Leftrightarrow m=2$ and $n=p$ that is Γ is a Star Graph and $\log\left(\left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil\right) = E(\Gamma, SO_{rd}(\Gamma)) \Leftrightarrow$ $(m-1) = \left\lfloor \frac{p}{2} \right\rfloor$ and $(n-1) = \left\lceil \frac{p}{2} \right\rceil$ that is Γ is a complete Bipartite Graph. Hence,

$$\log(p-1) \leq E(\Gamma, SO_{rd}(\Gamma)) \leq \log\left(\left\lfloor \frac{p}{2} \right\rfloor \left\lceil \frac{p}{2} \right\rceil\right)$$

□

Theorem 3.5. *Suppose Γ is a chemical graph having p-vertices, then*

$$E(\Gamma, SO_{rd}(\Gamma)) \leq \log(SO_{rd}(\Gamma)) - \log 3$$

Proof. In any chemical graph Γ 4 is the maximum degree of a vertex and 1 is the minimum degree of the vertex.

$$\begin{aligned}
E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log\sqrt{(d_s-1)^2+(d_r-1)^2} \\
&= \log SO_{rd}(\Gamma) - \frac{1}{2}\left(\log{(4-1)^2}+(1-1)^2\right) \\
&= \log SO_{rd}(\Gamma) - \frac{1}{2}\log(9) \\
&= \log SO_{rd}(\Gamma) - \frac{1}{2}\log 9 \\
&= \log SO_{rd}(\Gamma) - \log\sqrt{9} \\
E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log 3
\end{aligned}$$

Hence,

$$E(\Gamma, SO_{rd}(\Gamma)) \leq \log(So_{rd}(\Gamma)) - \log 3$$

□

Theorem 3.6. *Suppose Γ is a complete graph having p-vertices, then*

$$E(\Gamma, SO_{rd}(\Gamma)) \leq \frac{p(p-1)(p-2)\sqrt{2}}{2} - \log\sqrt{2}(p-2)$$

Proof. In any Complete Graph of order p. The $\phi = p-1$ and $\theta = p-1$

$$E(\Gamma, SO_{rd}(\Gamma)) \leq \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2}$$

$$
\begin{aligned}
SO_{rd}(\Gamma) &= \frac{p(p-1)(p-2)\sqrt{2}}{2} \\
&= \frac{p(p-1)(p-2)\sqrt{2}}{2} - \frac{1}{2}\log\left((p-1-1)^2+(p-1-1)^2\right) \\
&= \frac{p(p-1)(p-2)\sqrt{2}}{2} - \frac{1}{2}\log 2((p-2)^2) \\
&= \frac{p(p-1)(p-2)\sqrt{2}}{2} - \log\sqrt{2}((p-2)) \\
E(\Gamma, SO_{rd}(\Gamma)) &\leq \frac{p(p-1)(p-2)\sqrt{2}}{2} - \log\sqrt{2}(p-2)
\end{aligned}
$$

□

Theorem 3.7. *Let Γ be any tree having p-vertices. Then ,*

$$
E(\Gamma, SO_{rd}(\Gamma)) \leq (p-1)(p-2) - \log(p-1)
$$

Proof. Among all Trees of order $p \geq 4$, $\phi = 1$ and $\theta = p$

$$
\begin{aligned}
SO_{rd}(\Gamma) &= (p-1)(p-2) \\
E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\
&= (p-1)(p-2) - \frac{1}{2}\log\left((p-1)^2+(1-1)^2\right) \\
&= (p-1)(p-2) - \frac{1}{2}\log(p-1)^2 \\
&= (p-1)(p-2) - \log\sqrt{(p-1)^2} \\
E(\Gamma, SO_{rd}(\Gamma)) &\leq (p-1)(p-2) - \log(p-1)
\end{aligned}
$$

□

Theorem 3.8. *In a unicyclic graph*

$$
E(\Gamma, SO_{rd}(\Gamma)) \leq p\sqrt{2} + qp\sqrt{(q-1)^2+(p-1)^2} - (\log\sqrt{2} + \log\sqrt{(q-1)^2+(p-1)^2})
$$

Proof. In a unicyclic graph, a graph which contain exactly one cycle, $p \geq 3$ and $q, p \geq 2$. Then $U_p = C_p + K_{q,p}$ in cycle graph C_p the ϕ of a vertex and θ of a vertex is 2 and in complete Bipartite graph $K_{q,p}$ the ϕ of a vertex is $q+p$ and θ degree is $q+p$.

$$
\begin{aligned}
E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2)} \\
SO_{rd}(\Gamma) &= p\sqrt{2} + qp\sqrt{(q-1)^2+(p-1)^2} \\
&= p\sqrt{2} + qp\sqrt{(q-1)^2+(p-1)^2} - \\
&(\log\sqrt{(2-1)^2+(2-1)^2} + \log\sqrt{(q-1)^2+(p-1)^2} \\
&= p\sqrt{2} + qp\sqrt{(q-1)^2+(p-1)^2} - \\
&(\log\sqrt{1^2+1^2} + \log\sqrt{(q-1)^2+(p-1)^2)} \\
&= p\sqrt{2} + qp\sqrt{(q-1)^2+(p-1)^2} - \\
&(\log\sqrt{2} + \log\sqrt{(q-1)^2+(p-1)^2)} \\
E(\Gamma, SO_{rd}(\Gamma)) &\leq p\sqrt{2} + qp\sqrt{(q-1)^2+(p-1)^2} - (\log\sqrt{2} + \log\sqrt{(q-1)^2+(p-1)^2)}
\end{aligned}
$$

□

Theorem 3.9. *For a Star Graph*

$$
E(\Gamma, SO_{rd}(\Gamma)) \leq p(p-1) - \log(p-1)
$$

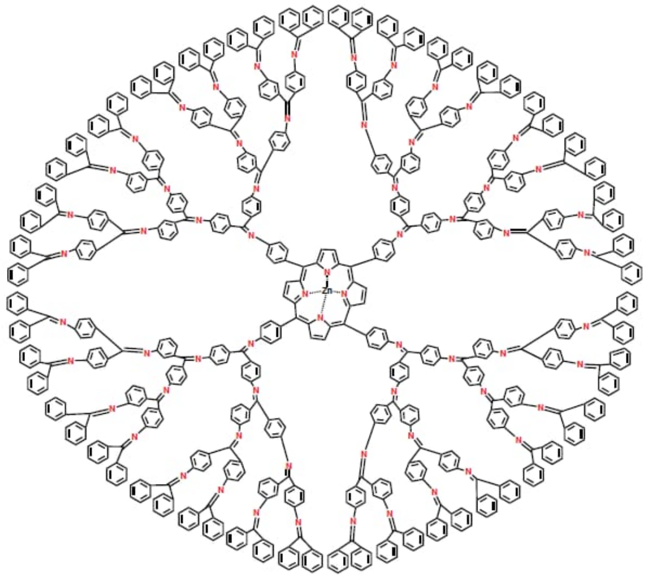

Figure 1: Porphyrin Dendrimer

Proof. For Star Graph, $\phi = 1$ and $\theta = p$

$$\begin{aligned} SO_{rd}(\Gamma) &= p(p-1) \\ E(\Gamma, SO_{rd}(\Gamma)) &\leq \log SO_{rd}(\Gamma) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\ &= p(p-1) - \log\sqrt{((p-1)^2+(1-1)^2)} \\ &= p(p-1) - \log\sqrt{(p-1)^2} \\ E(\Gamma, SO_{rd}(\Gamma)) &\leq p(p-1) - \log(p-1) \end{aligned}$$

□

4. Applications in Fractals

Example 4.1. *Consider the porphyrin dendrimers shown in Figure 1. We denote the graph of porphyrin dendrimers by Γ, and the edge partition of Γ is given in Table 1. Using Table 1 and definition.*

We have the following entropies for porphyrin dendrimers:

$$\begin{aligned} E(\Gamma, SO_{rd}) &= \log(SO_{rd}) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\ &= \log(263.0873614n - 20.48747568) - \left[\mid E_1 \mid \log\sqrt{4} + \mid E_2 \mid \log\sqrt{9} + \mid E_3 \mid \log\sqrt{2}\right] - \\ &\left[\mid E_4 \mid \log\sqrt{5} + \mid E_5 \mid \log\sqrt{8} + \mid E_6 \mid \log\sqrt{13}\right] \\ &= \log(263.0873614n - 20.48747568) - \left[(2n)\log\sqrt{4} + (24n)\log\sqrt{9} + (10n-5)\log\sqrt{2}\right] \\ &- \left[(48n-6)\log\sqrt{5} + (13n)\log\sqrt{8} + (8n)\log\sqrt{13}\right] \\ &= \log(263.0873614n - 20.48747568) - \log(40.65925258n - 2.849) \\ &= \log(SO_{rd}) - \log(40.65925258n - 2.849) \end{aligned}$$

Table 1: Edge partition of porphyrin dendrimer

(d_r, d_s)	(1,3)	(1,4)	(2,2)	(2,3)	(3,3)	(3,4)
Number of edges	$2n$	$24n$	$(10n-5)$	$48n-6$	$13n$	$8n$

Example 4.2. *The graph G of zinc-porphyrin dendrimer is shown in Figure 2, and the edge partition for this dendrimer is given in Table 2. We have the following computations for the entropies of zinc-porphyrin dendrimer:*

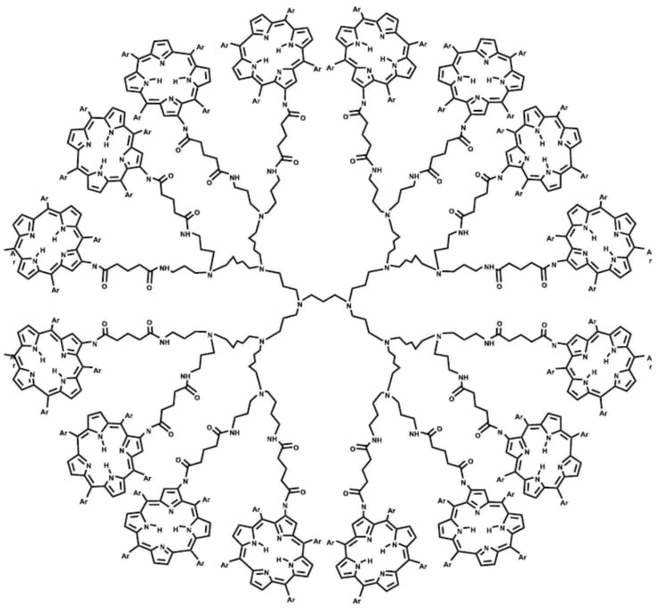

Figure 2: Zinc Porphyrin Dendrimer

$$
\begin{aligned}
E(\Gamma, SO_{rd}) &= \log(SO_{rd}) - \log\sqrt{(d_r-1)^2+(d_s-1)^2} \\
&= \log(2^n(134.6975531) - 101.0110981) - \Big[\mid E_1 \mid \log\sqrt{2} + \mid E_2 \mid \log\sqrt{5} + \mid E_3 \mid \log\sqrt{8} + \mid E_4 \mid \log\sqrt{13}\Big] \\
&= \log(2^n(134.6975531) - 101.0110981) - \Big[(16.2^n-4)\log\sqrt{2} + (40.2^n-16)\log\sqrt{5} + (8.2^n-16)\log\sqrt{8} + (4)\log\sqrt{13}\Big] \\
&= \log(2^n(134.6975531) - 101.0110981) - \log(20.2^n - 15.64642663) \\
&= \log(SO_{rd}) - \log(20.2^n - 15.64642663)
\end{aligned}
$$

Table 2: Edge partition of zinc-porphyrin dendrimers

(d_r, d_s)	(2,2)	(2,3)	(3,3)	(3,4)
Number of edges	(16.2^n-4)	(40.2^n-16)	(8.2^n-16)	4

5. Conclusions

Weighted entropy is a generalization of Shannon's entropy and is measure of information supplied by a probabilistic experiments whose elementary events are characterized both by their objective probabilities and by some qualitative weights. It is useful to rank chemicals and may be used to balance the amount of information. Weighted entropy also found applications in coding throey. In this paper we have studied weighted entropy with sombor reduced index SO_{rd} and prove results by using different graphs. Our next aim is to work on entropy of weighted graphs with geometric and sum connectivity edge weights. It would be interesting to work on entropy of weighted graphs with some other degree- and distance-based topological indices. The bounds of degree-based network entropy can also be used in national security, Internet networks, social networks, structural chemistry, ecological networks, computational systems biology, etc. They will play an important role in analyzing structural symmetry and asymmetry in real networks in the future.

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