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Caputo Fractional Derivative Inequalities for modified $(h, m)-$Convex Functions

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Abstract

Fractional calculus has emerged as a powerful tool in various branches of science and engineering, including mathematical modeling of complex phenomena. In particular, the Caputo k-fractional derivative has been extensively used to model various real-world problems. In this paper, we focus on developing Hadamard type inequalities for modified $(h, m)-$convex functions via the Caputo $k-$fractional derivatives. The main objective of this paper is to provide a new approach to estimating the fractional derivative of modified $(h, m)-$convex functions through the use of two integral identities involving the nth order derivatives of given functions. The results obtained in this paper can have significant applications in various fields of engineering and physics, including the modeling of complex systems governed by fractional differential equations.

Keywords: Caputo derivatives, Modified $(h, m)-$convex function, Hadamard inequality.

Fractional calculus is a branch of mathematics that deals with the generalization of ordinary calculus to non-integer orders of differentiation and integration. It provides a powerful tool for modeling and analyzing complex systems in various fields of science and engineering, including physics, chemistry, biology, economics, and signal processing. The concept of fractional calculus dates back to the 17th century, where Leibniz and L'Hopital independently studied fractional differentiation and integration. However, the field has only recently gained significant attention due to its ability to model many real-world phenomena more accurately than traditional integer-order calculus.

Fractional calculus has a wide range of applications, including the modeling of anomalous diffusion, viscoelasticity, and control of complex systems. It has also been used to solve fractional differential equations, which have been found to be more effective than traditional differential equations in modeling real-world problems.

The basic concepts of fractional calculus include fractional derivatives, fractional integrals, and fractional differential equations. Fractional derivatives and integrals generalize the traditional concepts of differentiation and integration to non-integer orders, while fractional differential equations are equations that involve fractional derivatives. In summary, fractional calculus is a rapidly growing field that has gained significant attention due to its applications in modeling and analyzing complex systems. Convex functions are an important concept in mathematical analysis and optimization. They play a crucial role in a wide range of fields, including economics, engineering, physics, and computer science.

A function is said to be convex if it satisfies the property that the line segment joining any two points on the graph of the function lies above the graph. This can be expressed mathematically as the inequality:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2),$$

where x_1 and x_2 are two points on the graph of the function, λ is a parameter between 0 and 1, and f is the convex function.

Convex functions have a number of important properties that make them useful in optimization problems. For example, any local minimum of a convex function is also a global minimum, and the set of global minima of a convex function is always a convex set. This property makes it easier to find the optimal solution in many optimization problems. Convex functions have a wide range of applications in different areas of science and engineering. In economics, convexity is used to model production functions and utility functions. In physics, it is used to study the behavior of energy and potential functions. In computer science, it is used to design efficient algorithms for optimization problems. In summary, convex functions are an important mathematical concept that have many applications in science and engineering. The study of convex functions is an important part of optimization theory and has led to the development of many powerful optimization algorithms.

Inequalities for convex functions are a fundamental topic in mathematical analysis and optimization. These inequalities are used to bound the values of convex functions, and they play an essential role in a wide range of applications, including

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economics, engineering, physics, and computer science. The most well-known inequality for convex functions is the Jensen's inequality, which states that the value of a convex function of a weighted average of points is always less than or equal to the weighted average of the function values at those points. This inequality has been used in many areas, including probability theory, information theory, and statistics. Another important inequality for convex functions is the Hermite-Hadamard inequality, which provides an upper bound on the integral of a convex function over a closed interval. This inequality has been used in the study of partial differential equations, optimization, and numerical analysis.

In addition to these two inequalities, there are many other inequalities for convex functions, including the Young's inequality, the Holder's inequality, and the Minkowski's inequality. These inequalities have been used in various applications, such as signal processing, control theory, and image processing.

The study of inequalities for convex functions is a rich area of research that has led to the development of many powerful mathematical tools and techniques. These inequalities play a crucial role in the development of optimization algorithms and have practical applications in a wide range of fields

The Hermite-Hadamard inequality states that for a convex function $f : [a, b] \to \mathbb{R}$, the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In other words, the average value of a convex function f on an interval $[a, b]$ is greater than or equal to its value at the midpoint $\frac{a+b}{2}$, and less than or equal to the average of its values on the two subintervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$.

The aim of this paper is to apply the k-analog of the definition of the Caputo fractional derivative, which is called the Caputo k-fractional derivative [7], in order to establish Hadamard type inequalities for modified (h, m)-convex functions.

1. Definitions and Basic Results

In this section, we provide the necessary definitions and basic results for the subsequent analysis. We begin by introducing the concept of modified (h, m)-convex functions.

We start with the Caputo fractional derivative operator [6].

Definition 1.1. *Let f be a function in $AC^n[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. For $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, the Caputo fractional derivative of order α of f is defined as:*

$$^{C}D^{\alpha}a^{+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int a^{x}\frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, \qquad x > a, \tag{1}$$

$$^{C}D^{\alpha}b^{-}f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int x^{b}\frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, \qquad x < b, \tag{2}$$

where $n = [\alpha] + 1$, $f^{(n)}(t)$ denotes the nth derivative of f and Γ denotes the gamma function.

Note that if $\alpha = n \in 1, 2, 3, ...$ and the usual derivatives of order n exist, then $(^{C}D^{\alpha}a^{+}f)(x)$ coincides with $f^{(n)}(x)$, whereas $(^{C}D^{\alpha}b^{-}f)(x)$ coincides with $f^{(n)}(x)$, with exactness to a constant multiplier $(-1)^n$. Additionally, we have $(^{C}D^{0}a^{+}f)(x) = (^{C}D^{0}b^{-}f)(x) = f(x)$, where $n = 1$ and $\alpha = 0$.

The definition of Caputo $k-$fractional derivatives of order α given in [7] can be enhanced as follows:

Definition 1.2. *Let $f \in AC^n[a, b]$. The Caputo $k-$fractional derivatives of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ of f are defined by the formulas (1) and (2), where $k \geq 1$, $n = [\alpha] + 1$, and $\Gamma_k(\alpha)$ is the $k-$analogue of the gamma function, defined by*

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{\frac{-t^k}{k}}dt,$$

with the property $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$ for all $\alpha \in \mathbb{C}$.

It is worth noting that if $\alpha = n \in 1, 2, 3, ...$ and usual derivatives of order n exists, then the Caputo $k-$fractional derivatives $(^{C}D^{\alpha,1}a^{+}f)(x)$ and $(^{C}D^{\alpha,1}b^{-}f)(x)$ coincide with $f^{(n)}(x)$, except for a constant multiplier $(-1)^n$.

Definition 1.3 (Convex function). *A function $f : I \to \mathbb{R}$ is said to be convex if for any two points $x, y \in I$ and $\alpha \in [0, 1]$, the function satisfies the following condition:*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y),$$

which means that the function value at the midpoint of the interval between x and y is less than or equal to the average of the function values at x and y. In other words, the graph of a convex function lies below the chord that connects any two points on the graph.

Definition 1.4 (h-convex function)**.** *An h-convex function $f : J \to \mathbb{R}$ is a non-negative function that satisfies the following inequality for any two points $x, y \in J$ and $\alpha \in (0,1)$:*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y),$$

where h is a non-negative function defined on the interval $(0,1)$, which is called the weight function. In this case, the graph of an h-convex function lies below the weighted average of the function values at x and y, where the weights are given by $h(\alpha)$ and $h(1-\alpha)$.

Definition 1.5 (Modified h-convex function)**.** *A modified h-convex function $f : J \to \mathbb{R}$ is a non-negative function that satisfies the following inequality for any two points $x, y \in J$ and $\alpha \in [0,1]$:*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + (1-h(\alpha))f(y),$$

where h is a non-negative function defined on the interval $[0,1]$. In this case, the graph of a modified h-convex function lies below the convex combination of the function values at x and y, where the weight of $f(x)$ is given by $h(\alpha)$ and the weight of $f(y)$ is given by $1-h(\alpha)$.

Definition 1.6 (Modified $(h-m)$ convex function)**.** *A modified $(h-m)$ convex function $f : J \to \mathbb{R}$ is a non-negative function that satisfies the following inequality for any two points $x, y \in J$ and $\alpha, m \in [0,1]$:*

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + m(1-h(\alpha))f(y),$$

where h is a non-negative function defined on the interval $[0,1]$. In this case, the graph of a modified $(h-m)$ convex function lies below the weighted average of the function.

Modified (h,m) convex functions are a generalization of modified h-convex functions. In the definition of modified (h,m) convex functions, we have an additional parameter m which controls the influence of the second term of the inequality. When $m = 1$, we get the definition of modified h-convex functions as a special case. Therefore, modified h-convex functions are a particular case of modified (h,m) convex functions.

On the other hand, when $h(\alpha) = \alpha$, we get the definition of convex functions as a special case of modified (h,m) convex functions. Thus, convex functions can also be seen as a particular case of modified (h,m) convex functions.

Here are some basic properties of modified (h,m) convex functions:

- If f is a modified (h,m) convex function, then cf is also modified (h,m) convex for any non-negative constant c.
- If f and g are modified (h,m) convex functions, then $f+g$ is also modified (h,m) convex.
- If f and g are modified (h,m) convex functions and h is increasing, then $f \cdot g$ is also modified (h,m) convex.
- If f is a modified (h,m) convex function, then f is continuous on the interior of its domain.
- If f is a modified (h,m) convex function, then f is locally Lipschitz continuous on the interior of its domain.
- If f is a modified (h,m) convex function, then f is quasi-monotone on its domain. That is, for any $x, y \in I$, if $x < y$ and $f(x) \geq f(y)$, then $f(z) \geq f(y)$ for all $z \in (x,y)$.
- If f is a modified (h,m) convex function and $h(\alpha) \leq \alpha$, then f is (h,m) convex.
- If f is a modified (h,m) convex function on J and g is a modified (h,m) convex function on K, where J and K are non-empty, closed and bounded intervals with $m_J \geq 0$ and $m_K \geq 0$, then $f \oplus g$ defined as $f \oplus g(x,y) = f(x) + g(y) + m_J x + m_K y$ is a modified (h,m) convex function on $J \times K$.

2. Main Results

Theorem 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that f is n times absolutely continuous on $[a,b]$ with $0 \leq a < b$. Suppose that $f^{(n)}$ is a modified (h,m)-convex function on $[a,mb]$ with $m \in (0,1]$. Then the following inequality for Caputo k-fractional derivatives holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)({}^C D^{\alpha,k}a^+ f(mb) + (1-h\left(\frac{1}{2}\right))m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}b^- f(\frac{a}{m}))\right], \quad (3)$$

where $\alpha \in (0,k)$, and the constants Γ_k and ${}^C D^{\alpha,k}_{x^+}$ denote the Caputo k-gamma function and the Caputo k-fractional derivative with respect to the right endpoint x, respectively.

Furthermore, we have the following upper bound:

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq \left(n-\frac{\alpha}{k}\right)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right) f^{(n)}\left(\frac{a}{m^2}\right) + mh\left(\frac{1}{2}\right) f^{(n)}(b)\right)\int_0^1 (1-h(t))t^{n-\frac{\alpha}{k}-1}dt + \left(m\left(1-h\left(\frac{1}{2}\right)\right) f^{(n)}(b) + h\left(\frac{1}{2}\right) f^{(n)}(a)\right)\int_0^1 h(t)t^{n-\frac{\alpha}{k}-1}dt\right].$$

Here, h is a non-negative function and $h(1/2)=1/2$, $m\in(0,1]$, and α is the order of the Caputo k-fractional derivative.

Proof. Since $f^{(n)}$ is a modified $(h,m)-$convex function on $[a,mb]$, this means that for any $x,y\in[a,mb]$ and $t\in[0,1]$, we have the following inequality:

$$f^{(n)}\left(\frac{um+v}{2}\right) \leq h\left(\frac{1}{2}\right) f^{(n)}(v) + m\left(1-h\left(\frac{1}{2}\right)\right) f^{(n)}(u), \quad u,v\in[a,b]. \tag{4}$$

By substituting $u=(1-t)\frac{a}{m}+tb\leq b$ and $v=m(1-t)b+ta\geq a$ in the above inequality for $t\in[0,1]$ and multiplying both sides by $t^{n-\frac{\alpha}{k}-1}$, we obtain:

$$f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt \leq h\left(\frac{1}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}\left(m(1-t)b+ta\right)dt + m\left(1-h\left(\frac{1}{2}\right)\right)\int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}\left((1-t)\frac{a}{m}+tb\right)dt.$$

Now, if we set $w=(1-t)\frac{a}{m}+tb$ and $z=m(1-t)b+ta$ in the right hand side of the above inequality, we obtain

$$f^{(n)}\left(\frac{bm+a}{2}\right)\frac{1}{n-\frac{\alpha}{k}} \leq h\left(\frac{1}{2}\right)\int_a^{mb}\left(\frac{mb-z}{mb-a}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(z)dz}{(mb-a)} + m\left(1-h\left(\frac{1}{2}\right)\right)\int_{\frac{a}{m}}^{b}\left(\frac{w-\frac{a}{m}}{b-\frac{b}{m}}\right)^{n-\frac{\alpha}{k}-1}\frac{mf^{(n)}(w)dw}{(b-\frac{a}{m})}.$$

From the above expressions, we can write the following inequality:

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)({}^C D^{\alpha,k}_{a^+} f(mb)) + \left(1-h\left(\frac{1}{2}\right)\right) m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-} f(\frac{a}{m}))\right]. \tag{5}$$

By the modified (h,m)-convexity of $f^{(n)}$ on $[a,mb]$, we have for $t\in[0,1]$ and $w=(1-t)\frac{a}{m}+tb\leq b$ and $z=m(1-t)b+ta\geq a$:

$$mf^{(n)}\left((1-t)\frac{a}{m}+tb\right) \leq m^2(1-h(t))f^{(n)}\left(\frac{a}{m^2}\right) + mh(t)f^{(n)}(b). \tag{6}$$

Multiplying both sides of the inequality in (6) by $(n-\frac{\alpha}{k})(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, we obtain

$$\frac{m^{n-\frac{\alpha}{k}+1}\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}} k\Gamma_k(n-\frac{\alpha}{k}+k)(-1)^n({}^C D^{\alpha,k}_{b^-} f(\frac{a}{m})) \leq m\left(n-\frac{\alpha}{k}\right)\left(1-h\left(\frac{1}{2}\right)\right) \times\left[mf^{(n)}\left(\frac{a}{m^2}\right)\int_0^1(1-h(t))t^{n-\frac{\alpha}{k}-1}dt + f^{(n)}(b)\int_0^1 h(t)t^{n-\frac{\alpha}{k}-1}dt\right]. \tag{7}$$

Modified $(h,m)-$ convexity of $f^{(n)}$ implies that for $x,y\in[a,mb]$ and $t\in[0,1]$, we have:

$$f^{(n)}\left(m(1-t)b+ta\right) \leq m(1-h(t))f^{(n)}(b) + h(t)f^{(n)}(a).$$

Multiplying both side by $(n-\frac{\alpha}{k})(h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\frac{h\left(\frac{1}{2}\right)}{(mb-a)^{n-\frac{\alpha}{k}}} k\Gamma_k(n-\frac{\alpha}{k}+k)({}^C D^{\alpha,k}_{a^+} f(mb)) \leq \left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right) \times\left[mf^{(n)}(b)\int_0^1(1-h(t))t^{n-\frac{\alpha}{k}-1}dt + f^{(n)}(a)\int_0^1 h(t)t^{n-\frac{\alpha}{k}-1}dt\right]. \tag{8}$$

Addition of (7) and (8) yields.

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[\left(1-h\left(\frac{1}{2}\right)\right) m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-} f(\frac{a}{m})) + h\left(\frac{1}{2}\right)({}^C D^{\alpha,k}_{a^+} f(mb)\right]$$

$$\leq \left(n-\frac{\alpha}{k}\right)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)+m\left(h\left(\frac{1}{2}\right)\right)f^{(n)}(b)\right)\int_0^1(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right.$$
$$\left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h(t)t^{n-\frac{\alpha}{k}-1}dt\right]. \quad (9)$$

By combining the inequalities (5) and (9), we get the required result.

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)({}^C D^{\alpha,k}_{a^+}f(mb)+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-}f(\frac{a}{m}))\right] \\ &\leq \left(n-\frac{\alpha}{k}\right)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)+mh\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right. \\ &\quad \left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h(t)t^{n-\frac{\alpha}{k}-1}dt\right]. \end{aligned}$$

□

Corollary 2.1. *By setting $k=1$ in the inequality (3), the following Caputo fractional derivatives inequality holds:*

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[h\left(\frac{1}{2}\right)({}^C D^{\alpha}_{a^+}f(mb)+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\alpha+1}(-1)^n({}^C D^{\alpha}_{b^-}f(\frac{a}{m}))\right] \\ &\leq (n-\alpha)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)+mh\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1(1-h(t))t^{n-\alpha-1}dt\right. \\ &\quad \left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h(t)t^{n-\alpha-1}dt\right]. \end{aligned} \quad (10)$$

Corollary 2.2. *By setting $m=1$ in the inequality (3), the following Caputo $k-$fractional derivatives inequality holds for modified $h-$convex functions:*

$$\begin{aligned} f^{(n)}\left(\frac{b+a}{2}\right) &\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)({}^C D^{\alpha,k}_{a^+}f(b)+\left(1-h\left(\frac{1}{2}\right)\right)(-1)^n({}^C D^{\alpha,k}_{b^-}f(a))\right] \\ &\leq \left(n-\frac{\alpha}{k}\right)\left[\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(a)+h\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1(1-h(t))t^{n-\frac{\alpha}{k}-1}dt\right. \\ &\quad \left.+\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h(t)t^{n-\frac{\alpha}{k}-1}dt\right]. \end{aligned} \quad (11)$$

Corollary 2.3. *By setting $k=1$ and $m=1$ in the inequality (3), the following Caputo fractional derivatives inequality holds:*

$$\begin{aligned} f^{(n)}\left(\frac{b+a}{2}\right) &\leq \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[h\left(\frac{1}{2}\right)({}^C D^{\alpha}_{a^+}f(b)+\left(1-h\left(\frac{1}{2}\right)\right)(-1)^n({}^C D^{\alpha}_{b^-}f(a))\right] \\ &\leq (n-\alpha)\left[\left(1-h\left(\frac{1}{2}\right)f^{(n)}(a)+h\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1(1-h(t))t^{n-\alpha-1}dt\right. \\ &\quad \left.+\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h(t)t^{n-\alpha-1}dt\right]. \end{aligned} \quad (12)$$

Corollary 2.4. *If we choose h is identity function in (3) function, the following Caputo $k-$fractional derivatives inequality hold:*

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{b^-}f(\frac{a}{m}))+({}^C D^{\alpha,k}_{a^+}f(mb)\right] \\ &\leq \frac{1}{2}\left[\left(n-\frac{\alpha}{k}\right)\left(m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\beta(2,n-\frac{\alpha}{k})\right. \\ &\quad \left.+\left(mf^{(n)}(b)+f^{(n)}(a)\right)\right]. \end{aligned} \quad (13)$$

Corollary 2.5. *If h is identity function and set m=1 in (3) the following Caputo $k-$fractional derivatives inequality hold for convex function:*

$$\begin{aligned} f^{(n)}\left(\frac{b+a}{2}\right) &\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[(-1)^n({}^C D^{\alpha,k}_{b^-}f(a)+({}^C D^{\alpha,k}_{a^+}f(b)\right] \\ &\leq \frac{1}{2}\left[\left(n-\frac{\alpha}{k}\right)\left(f^{(n)}(a)+f^{(n)}(b)\right)\beta(2,n-\frac{\alpha}{k})\right. \\ &\quad \left.+\left(f^{(n)}(b)+f^{(n)}(a)\right)\right]. \end{aligned} \quad (14)$$

Corollary 2.6. *If h is identity function and set k=1 in (3) the following Caputo fractional derivatives inequality hold for convex function:*

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right) &\leq \frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left[m^{n-\alpha+1}(-1)^n({}^CD^{\alpha}_{b^-}f(\frac{a}{m}))+({}^CD^{\alpha}_{a^+}f(mb)\right] \\ &\leq \frac{1}{2}\left[(n-\alpha)\left(m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\beta(2,n-\alpha)\right. \\ &\left.+\left(mf^{(n)}(b)+f^{(n)}(a)\right)\right]. \end{aligned} \tag{15}$$

Corollary 2.7. *If h is identity function and set k=1 and m=1 in (3) the following Caputo fractional derivatives inequality hold for convex function:*

$$\begin{aligned} f^{(n)}\left(\frac{b+a}{2}\right) &\leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[(-1)^n({}^CD^{\alpha}_{b^-}f(a))+({}^CD^{\alpha}_{a^+}f(b)\right] \\ &\leq \frac{1}{2}\left[(n-\alpha)\left(f^{(n)}(a)+f^{(n)}(b)\right)\beta(2,n-\alpha)\right. \\ &\left.+\left(f^{(n)}(b)+f^{(n)}(a)\right)\right]. \end{aligned} \tag{16}$$

Theorem 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,b], 0\leq a<b$. Also let $f^{(n)}$ be a modified $(h,m)-$convex function on $[a,mb]$ with $m\in(0,1]$. Then the following inequality for Caputo k-fractional derivatives hold:*

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right) &\leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)\right. \\ &\left.+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right] \\ &\leq \left(n-\frac{\alpha}{k}\right)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)\right.\right. \\ &\left.+mh\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}dt \\ &\left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\frac{\alpha}{k}-1}dt\right]. \end{aligned} \tag{17}$$

Proof. By putting $u=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $v=\frac{t}{2}a+m\frac{(2-t)}{2}b$ in (4) where $t\in[0,1]$, and multiplying with $t^{n-\frac{\alpha}{k}-1}$, then integrating over $[0,1]$ one can have

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt \leq & h\left(\frac{1}{2}\right)\left[\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt\right] \\ &+m\left(1-h\left(\frac{1}{2}\right)\right)\left[\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right]. \end{aligned}$$

By change of variables, as we did to get (5), one can also get

$$\begin{aligned} f^{(n)}\left(\frac{bm+a}{2}\right) \leq & 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}} \\ &\times\left[h\left(\frac{1}{2}\right)\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m^2}\right)\right)\right]. \end{aligned} \tag{18}$$

On the other hand, by using modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$f^{(n)}\left(\frac{t}{2}a+m\left(\frac{2-t}{2}b\right)\right)\leq h\left(\frac{t}{2}\right)f^{(n)}(a)+m\left(1-h\left(\frac{t}{2}\right)\right)f^{(n)}(b).$$

Multiplying both side by $(n-\frac{\alpha}{k})(h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\begin{aligned} &\frac{2^{(n-\frac{\alpha}{k})}}{(mb-a)^{n-\frac{\alpha}{k}}}h\left(\frac{1}{2}\right)k\Gamma_k(n-\frac{\alpha}{k}+k)({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb))\leq\left(n-\frac{\alpha}{k}\right)\left(h\left(\frac{1}{2}\right)\right) \\ &\times\left[f^{(n)}(a)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\frac{\alpha}{k}-1}dt+mf^{(n)}(b)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}dt\right]. \end{aligned} \tag{19}$$

By using modified $(h,m)-$ convexity of $f^{(n)}$, we have

$$mf^{(n)}\left(\frac{t}{2}b+m\left(\frac{2-t}{2}\right)\frac{a}{m^2}\right)\leq mh\left(\frac{t}{2}\right)f^{(n)}(b)+m^2\left(1-h\left(\frac{t}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right).$$

Multiplying both side by $(n-\frac{\alpha}{k})(1-h(\frac{1}{2}))t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculation we get

$$\frac{2^{(n-\frac{\alpha}{k})}m^{n-\frac{\alpha}{k}+1}\left(1-h\left(\frac{1}{2}\right)\right)}{(mb-a)^{n-\frac{\alpha}{k}}}k\Gamma_k(n-\frac{\alpha}{k}+k)(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m^2}\right)\right)\leq m\left(n-\frac{\alpha}{k}\right)\left(1-h\left(\frac{1}{2}\right)\right)$$
$$\times\left[f^{(n)}(b)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\frac{\alpha}{k}-1}dt+mf^{(n)}\left(\frac{a}{m^2}\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}dt\right]. \tag{20}$$

Addition of (19) and (20) yields.

$$\begin{aligned}&2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)\right.\\&\left.+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\&\leq\left(n-\frac{\alpha}{k}\right)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)\right.\right.\\&\left.+mh\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}dt\\&\left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\frac{\alpha}{k}-1}dt\right].\end{aligned} \tag{21}$$

By combining the inequalities (18) and (21), we get the required result.

$$\begin{aligned}f^{(n)}\left(\frac{bm+a}{2}\right)&\leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)\right.\\&\left.+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\&\leq\left(n-\frac{\alpha}{k}\right)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)\right.\right.\\&\left.+mh\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}dt\\&\left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\frac{\alpha}{k}-1}dt\right].\end{aligned} \tag{22}$$

□

Corollary 2.8. *By setting $k=1$ in inequality (22), the following inequality holds for caputo fractional derivatives:*

$$\begin{aligned}f^{(n)}\left(\frac{bm+a}{2}\right)&\leq 2^{(n-\alpha)}\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[h\left(\frac{1}{2}\right)\left({}^{C}D^{\alpha}_{(\frac{a+bm}{2})^+}f(mb)\right)\right.\\&\left.+\left(1-h\left(\frac{1}{2}\right)\right)m^{n-\alpha+1}(-1)^n\left({}^{C}D^{\alpha}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]\\&\leq(n-\alpha)\left[\left(m^2\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}\left(\frac{a}{m^2}\right)\right.\right.\\&\left.+mh\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\alpha-1}dt\\&\left.+\left(m\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\alpha-1}dt\right].\end{aligned} \tag{23}$$

Corollary 2.9. *By setting $m=1$ in inequality (22), the following Caputo $k-$fractional derivatives holds:*

$$\begin{aligned}f^{(n)}\left(\frac{b+a}{2}\right)&\leq 2^{(n-\frac{\alpha}{k})}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\left[h\left(\frac{1}{2}\right)\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)\right)+\left(1-h\left(\frac{1}{2}\right)\right)(-1)^n\left({}^{C}D^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)\right)\right]\\&\leq\left(n-\frac{\alpha}{k}\right)\left[\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(a)+h\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\frac{\alpha}{k}-1}dt\right.\\&\left.+\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\frac{\alpha}{k}-1}dt\right].\end{aligned} \tag{24}$$

Corollary 2.10. *By setting $m=1$ and $k=1$ in inequality (22), the following inequality holds for convex function via Caputo fractional derivatives:*

$$f^{(n)}\left(\frac{b+a}{2}\right)\leq 2^{(n-\alpha)}\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[h\left(\frac{1}{2}\right)\left({}^{C}D^{\alpha}_{(\frac{a+b}{2})^+}f(b)\right)+\left(1-h\left(\frac{1}{2}\right)\right)(-1)^n\left({}^{C}D^{\alpha}_{(\frac{a+b}{2})^-}f(a)\right)\right]$$

$$\leq (n-\alpha)\left[\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(a)+h\left(\frac{1}{2}\right)f^{(n)}(b)\right)\int_0^1\left(1-h\left(\frac{t}{2}\right)\right)t^{n-\alpha-1}dt\right.$$
$$\left.+\left(\left(1-h\left(\frac{1}{2}\right)\right)f^{(n)}(b)+h\left(\frac{1}{2}\right)f^{(n)}(a)\right)\int_0^1 h\left(\frac{t}{2}\right)t^{n-\alpha-1}dt\right]. \tag{25}$$

Corollary 2.11. *If we choose h is identity function in (22), the following Caputo k−fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq 2^{(n-\frac{\alpha}{k}-1)}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\frac{\alpha}{k}+1}(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]$$
$$\leq \frac{1}{4}\left(n-\frac{\alpha}{k}\right)\left[\left(m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\left(\frac{(nk-\alpha+2k)k}{(nk-\alpha)(nk-\alpha-k)}\right)+\frac{\left(mf^{(n)}(b)+f^{(n)}(a)\right)k}{(nk-\alpha+k)}\right] \tag{26}$$

Corollary 2.12. *If we choose h is identity function k = 1 in (22), the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{bm+a}{2}\right) \leq 2^{(n-\alpha-1)}\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[\left({}^CD^{\alpha}_{(\frac{a+bm}{2})^+}f(mb)\right)+m^{n-\alpha+1}(-1)^n\left({}^CD^{\alpha}_{(\frac{a+bm}{2m})^-}f\left(\frac{a}{m}\right)\right)\right]$$
$$\leq \frac{1}{4}(n-\alpha)\left[\left(m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right)\left(\frac{(n-\alpha+2)}{(n-\alpha)(n-\alpha-1)}\right)+\frac{\left(mf^{(n)}(b)+f^{(n)}(a)\right)}{(nk-\alpha+1)}\right]. \tag{27}$$

Corollary 2.13. *If we choose h is identity function and m = 1 in (22), the following Caputo k−fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{b+a}{2}\right) \leq 2^{(n-\frac{\alpha}{k}-1)}\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}\left[\left({}^CD^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)\right)+(-1)^n\left({}^CD^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)\right)\right]$$
$$\leq \frac{1}{4}\left(n-\frac{\alpha}{k}\right)\left[\left(f^{(n)}(a)+f^{(n)}(b)\right)\left(\frac{(nk-\alpha+2k)k}{(nk-\alpha)(nk-\alpha-k)}\right)+\frac{\left(f^{(n)}(b)+f^{(n)}(a)\right)k}{(nk-\alpha+k)}\right]. \tag{28}$$

Corollary 2.14. *If we choose h is identity function and m = 1 and k = 1 in (22), the following Caputo fractional derivatives inequality holds:*

$$f^{(n)}\left(\frac{b+a}{2}\right) \leq 2^{(n-\alpha-1)}\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[\left({}^CD^{\alpha}_{(\frac{a+b}{2})^+}f(b)\right)+(-1)^n\left({}^CD^{\alpha}_{(\frac{a+b}{2})^-}f(a)\right)\right]$$
$$\leq \frac{1}{4}(n-\alpha)\left[\left(f^{(n)}(a)+f^{(n)}(b)\right)\left(\frac{(n-\alpha+2)}{(n-\alpha)(n-\alpha-1)}\right)+\frac{\left(f^{(n)}(b)+f^{(n)}(a)\right)}{(n-\alpha+1)}\right]. \tag{29}$$

Theorem 2.3. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,b], 0\leq a<b$. Also let $f^{(n)}$ be a modified $(h,m)-$convex function on $[a,mb]$ with $m\in(0,1]$. Then the following inequality for Caputo k-fractional derivatives hold:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\}$$
$$\leq\frac{mk}{nk-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]+\left[f^{(n)}(a)+f^{(n)}(b)\right.$$
$$\left.-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt.$$
$$\leq\frac{mk}{nk-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]+\left[f^{(n)}(a)+f^{(n)}(b)\right.$$
$$\left.-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]\frac{\left(\int_0^1(h(t))^q\right)^{\frac{1}{q}}}{(np-\frac{\alpha p}{k}-p+1)^{\frac{1}{p}}}. \tag{30}$$

where $p^{-1}+q^{-1}=1$ and $p>1$.

Proof. Since $f^{(n)}$ is modified $(h,m)-$convex function on $[a,mb]$ then for $m\in(0,1]$ and $t\in[0,1]$, we have

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)\leq h(t)\left[f^{(n)}(a)+f^{(n)}(b)\right]+m(1-h(t))\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right].$$

By multiplying both side of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating the above inequality with respect to t on $[0,1]$, we have

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}\{f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)\}$$

$$\leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt$$
$$+m\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}(1-h(t))dt.$$

If we set $x=ta+(1-t)b$ in the left side of above inequality, we get the following inequality

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\}$$
$$\leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt$$
$$+m\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}(1-h(t))dt.$$

After some calculations the above inequality becomes

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\}$$
$$\leq \frac{mk}{nk-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]$$
$$+\left[f^{(n)}(a)+f^{(n)}(b)-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]$$
$$\times\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt. \tag{31}$$

Thus, we get the first inequality of (30). The second inequality of (30) follows from the fact by using the Hölder inequality

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt \leq \frac{\left(\int_0^1(h(t))^q\right)^{\frac{1}{q}}}{(np-\frac{\alpha p}{k}-p+1)^{\frac{1}{p}}}, \tag{32}$$

and from (32) and (31) we get the required result.

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\}$$
$$\leq \frac{mk}{nk-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]+\left[f^{(n)}(a)+f^{(n)}(b)\right.$$
$$\left.-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt.$$
$$\leq \frac{mk}{nk-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]+\left[f^{(n)}(a)+f^{(n)}(b)\right.$$
$$\left.-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]\frac{\left(\int_0^1(h(t))^q\right)^{\frac{1}{q}}}{(np-\frac{\alpha p}{k}-p+1)^{\frac{1}{p}}}.$$

□

Corollary 2.15. *By setting $k=1$ in inequality (30), the following inequality holds via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha}_{b^-}f(a))\}$$
$$\leq \frac{m}{n-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]+\left[f^{(n)}(a)+f^{(n)}(b)\right.$$
$$\left.-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]\int_0^1 t^{n-\alpha-1}h(t)dt.$$
$$\leq \frac{m}{n-\alpha}\left[f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right]+\left[f^{(n)}(a)+f^{(n)}(b)\right.$$
$$\left.-m\left(f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)\right)\right]\frac{\left(\int_0^1(h(t))^q\right)^{\frac{1}{q}}}{(np-\frac{\alpha p}{k}-p+1)^{\frac{1}{p}}}. \tag{33}$$

Corollary 2.16. *By setting $m=1$ in inequality (30), the following inequality holds via Caputo k-fractional derivatives:*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}\{({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\}\leq \frac{k}{nk-\alpha}\left[f^{(n)}(a)+f^{(n)}(b)\right]. \tag{34}$$

Corollary 2.17. *By setting $m = 1$ and $k = 1$ in inequality (30), the following inequality holds via Caputo fractional derivatives:*

$$\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\{({}^C D^{\alpha}_{a^+} f(b)) + (-1)^n({}^C D^{\alpha}_{b^-} f(a))\} \leq \frac{1}{n-\alpha}\left[f^{(n)}(a) + f^{(n)}(b)\right]. \tag{35}$$

3. Caputo k-fractional Derivatives for functions whose nth derivative in absolute value are modified (h, m)-convex

The following lemma is helpful to prove the next result.

Lemma 3.1. *[24] Let $f : [a, mb] \to \mathbb{R}$ be a differentiable mapping on interval (a, mb) with $a \leq mb$.If $f \in C^{n+1}[a.mb]$, then the following equality for Caputo k-fractional integrals holds*

$$\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n({}^C D^{\alpha,k}_{mb^-} f(a))\right)$$
$$= \frac{mb-a}{2}\int_0^1 ((1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}) f^{(n+1)}(m(1-t)b + ta)dt.$$

Theorem 3.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a, b]$. If $|f^{(n+1)}|$ is modified (h,m)-convex function on [a,mb] with $m \in (0, 1]$. Then the following inequality Caputo for k-fractional derivatives holds:*

$$\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n({}^C D^{\alpha,k}_{mb^-} f(a))\right)\right|$$
$$\leq \frac{mb-a}{2}\left\{2m\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\frac{\alpha}{k}-n-1}}{n-\frac{\alpha}{k}+1}\right) + \frac{\left(\left|f^{(n+1)}(a) - mf^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\right.$$
$$\left.\times\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}} - (2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\left[\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (h(t))^q\right)^{\frac{1}{q}}\right]\right\}, \tag{36}$$

where $p^{-1} + q^{-1} = 1$.

Proof. From lemma 3.1 and by using the property of modulus, we get

$$\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n({}^C D^{\alpha,k}_{mb^-} f(a))\right)$$
$$\leq \frac{mb-a}{2}\int_0^1 |(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}|\left|f^{(n+1)}(m(1-t)b + ta)\right|dt.$$

By modified (h,m)-convexity of $|f^{(n+1)}|$, we have

$$\left|\frac{f^{(n)}(mb) + f^{(n)}(a)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+} f(mb)) + (-1)^n({}^C D^{\alpha,k}_{mb^-} f(a))\right)\right|$$
$$\leq \frac{mb-a}{2}\left\{\int_0^{\frac{1}{2}}((1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}})\left(m(1-h(t))|f^{(n+1)}(b)| + h(t)|f^{(n+1)}(a)|\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1((1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}})\left(m(1-h(t))|f^{(n+1)}(b)| + h(t)|f^{(n+1)}(a)|\right)dt\right\}$$
$$= \frac{mb-a}{2}\left\{\left|f^{(n+1)}(a)\right|\left(\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(t)dt - \int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(t)dt\right)\right.$$
$$+m\left|f^{(n+1)}(b)\right|\left(\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}(1-h(t))dt - \int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}(1-h(t))dt\right)$$
$$+\left|f^{(n+1)}(a)\right|\left(\int_{\frac{1}{2}}^1 t^{n-\frac{\alpha}{k}}h(t)dt - \int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}}h(t)dt\right)$$
$$\left.+m\left|f^{(n+1)}(b)\right|\left(\int_{\frac{1}{2}}^1 t^{n-\frac{\alpha}{k}}(1-h(t))dt - \int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}}(1-h(t))dt\right)\right\}$$
$$= \frac{mb-a}{2}\left\{2m\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\frac{\alpha}{k}-n-1}}{n-\frac{\alpha}{k}+1}\right)\right.$$

$$+\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(t)dt\right)$$
$$-\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(t)dt\right)$$
$$+\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\int_{\frac{1}{2}}^{1}t^{n-\frac{\alpha}{k}}h(t)dt\right)$$
$$\left.-\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\int_{\frac{1}{2}}^{1}(1-t)^{n-\frac{\alpha}{k}}h(t)dt\right)\right\}.$$

Now, by using the Hölder's inequality in the right hand side of above inequality, we get

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{2m\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\frac{\alpha}{k}-n-1}}{n-\frac{\alpha}{k}+1}\right)\right.$$
$$+\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\frac{1-2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}}$$
$$-\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\frac{2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}}$$
$$+\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\frac{1-2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(h(t))^q\right)^{\frac{1}{q}}$$
$$\left.-\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)\left(\frac{2^{\frac{\alpha p}{k}-np-1}}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(h(t))^q\right)^{\frac{1}{q}}\right\}.$$

After some calculation we get thr desired result.

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{2m\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\frac{\alpha}{k}-n-1}}{n-\frac{\alpha}{k}+1}\right)+\frac{\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\right.$$
$$\left.\times\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\left[\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(h(t))^q\right)^{\frac{1}{q}}\right]\right\}.$$

□

Corollary 3.1. *By setting k=1 in inequality (36), the following Caputo fractional derivative inequality holds:*

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(({}^CD^{\alpha}_{a^+}f(mb))+(-1)^n({}^CD^{\alpha}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{mb-a}{2}\left\{2m\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\alpha-n-1}}{n-\alpha+1}\right)+\frac{\left(\left|f^{(n+1)}(a)-mf^{(n+1)}(b)\right|\right)}{(np-\alpha p+1)^{\frac{1}{p}}}\right.$$
$$\left.\times\left((1-2^{\alpha p-np-1})^{\frac{1}{p}}-(2^{\alpha p-np-1})^{\frac{1}{p}}\right)\left[\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(h(t))^q\right)^{\frac{1}{q}}\right]\right\}.$$

Corollary 3.2. *By setting m=1 in inequality (36), the following Caputo k-fractional derivative inequality holds:*

$$\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left(({}^CD^{\alpha,k}_{a^+}f(b))+(-1)^n({}^CD^{\alpha,k}_{b^-}f(a))\right)\right|$$
$$\leq\frac{b-a}{2}\left\{2\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\frac{\alpha}{k}-n-1}}{n-\frac{\alpha}{k}+1}\right)+\frac{\left(\left|f^{(n+1)}(a)-f^{(n+1)}(b)\right|\right)}{(np-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\right.$$
$$\left.\times\left((1-2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}-(2^{\frac{\alpha p}{k}-np-1})^{\frac{1}{p}}\right)\left[\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(h(t))^q\right)^{\frac{1}{q}}\right]\right\}.$$

Corollary 3.3. *By setting k=1 and m=1 in inequality (36), the following holds for convex function via Caputo fractional derivative:*

$$\left|\frac{f^{(n)}(b)+f^{(n)}(a)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(({}^{C}D^{\alpha}_{a^+}f(b))+(-1)^n({}^{C}D^{\alpha}_{b^-}f(a))\right)\right|$$
$$\leq\frac{b-a}{2}\left\{2m\left|f^{(n+1)}(b)\right|\left(\frac{1-2^{\alpha-n-1}}{n-\alpha+1}\right)+\frac{\left(\left|f^{(n+1)}(a)-f^{(n+1)}(b)\right|\right)}{(np-\alpha p+1)^{\frac{1}{p}}}\right.$$
$$\left.\times\left((1-2^{\alpha p-np-1})^{\frac{1}{p}}-(2^{\alpha p-np-1})^{\frac{1}{p}}\right)\left[\left(\int_0^{\frac{1}{2}}(h(t))^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(h(t))^q\right)^{\frac{1}{q}}\right]\right\}.$$

Lemma 3.2. *[24] Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on interval (a,mb) with $a\leq mb$. If $f\in C^{n+2}[a.mb]$, then the following equality for Caputo k-fractional integrals holds:*

$$\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)$$
$$=\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}f^{(n+2)}(ta+m(1-t)b)dt.$$

Theorem 3.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^{n+2}[a,b]$. If $|f^{(n+2)}|$ is modified (h,m)-convex function on [a,mb] with $m\in(0,1]$. Then the following inequality Caputo for k-fractional derivatives holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left\{m\left|f^{(n+2)}(b)\right|\left(1-\frac{2}{p(n-\frac{\alpha}{k}+1)+1}\right)\right.$$
$$\left.+\left(\left|f^{(n+2)}(a)\right|-m\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^qdt\right)^{\frac{1}{q}}\right\}. \quad (37)$$

Proof. Using lemma 3.2 and modified (h,m)-convexity of $|f^{(n+2)}|$, we find ☐

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\left|f^{(n+2)}(ta+m(1-t)b)\right|dt$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}$$
$$\times\left(h(t)\left|f^{(n+2)}(a)\right|+m(1-h(t))\left|f^{(n+2)}(b)\right|\right)dt$$
$$=\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\int_0^1 1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}$$
$$\times\left(m\left|f^{(n+2)}(b)\right|+\left(\left|f^{(n+2)}(a)\right|-m\left|f^{(n+2)}(b)\right|\right)h(t)\right)dt$$
$$=\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left\{m\left|f^{(n+2)}(b)\right|\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})dt\right.$$
$$\left.+\left(\left|f^{(n+2)}(a)\right|-m\left|f^{(n+2)}(b)\right|\right)\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)dt\right\}.$$

Now, by using Hölder inequality, we have

$$\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)dt\leq\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^qdt\right)^{\frac{1}{q}}.$$

This implies

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left(({}^{C}D^{\alpha,k}_{a^+}f(mb))+(-1)^n({}^{C}D^{\alpha,k}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\frac{\alpha}{k}+1)}\left\{m\left|f^{(n+2)}(b)\right|\left(1-\frac{2}{p(n-\frac{\alpha}{k}+1)+1}\right)\right.$$

$$+\left(\left|f^{(n+2)}(a)\right|-m\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

Corollary 3.4. *By setting k=1 in inequality (37), the following Caputo fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left(({}^C D^{\alpha}_{a^+}f(mb))+(-1)^n({}^C D^{\alpha}_{mb^-}f(a))\right)\right|$$
$$\leq\frac{(mb-a)^2}{2(n-\alpha+1)}\Bigg\{m\left|f^{(n+2)}(b)\right|\left(1-\frac{2}{p(n-\alpha+1)+1}\right)$$
$$+\left(\left|f^{(n+2)}(a)\right|-m\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

Corollary 3.5. *By setting m=1 in inequality (37), the following Caputo k-fractional derivatives inequality holds:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(b-a)^{n-\frac{\alpha}{k}}}\left(({}^C D^{\alpha,k}_{a^+}f(b))+(-1)^n({}^C D^{\alpha,k}_{b^-}f(a))\right)\right|$$
$$\leq\frac{(b-a)^2}{2(n-\frac{\alpha}{k}+1)}\Bigg\{\left|f^{(n+2)}(b)\right|\left(1-\frac{2}{p(n-\frac{\alpha}{k}+1)+1}\right)$$
$$+\left(\left|f^{(n+2)}(a)\right|-\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

Corollary 3.6. *By setting k=1 and m=1 in inequality (37), the following inequality holds for convex function via Caputo fractional derivatives:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left(({}^C D^{\alpha}_{a^+}f(b))+(-1)^n({}^C D^{\alpha}_{b^-}f(a))\right)\right|$$
$$\leq\frac{(b-a)^2}{2(n-\alpha+1)}\Bigg\{\left|f^{(n+2)}(b)\right|\left(1-\frac{2}{p(n-\alpha+1)+1}\right)$$
$$+\left(\left|f^{(n+2)}(a)\right|-\left|f^{(n+2)}(b)\right|\right)\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^q dt\right)^{\frac{1}{q}}\Bigg\}.$$

4. Conclusions

The paper "Hadamard-type inequalities for modified $(h,m)-$convex functions via Caputo $k-$fractional derivatives" presents an important contribution to the field of fractional calculus. Fractional calculus has become a powerful tool in various fields of science and engineering, especially in the modeling of complex phenomena. In particular, the Caputo $k-$fractional derivative has been widely used to model various real-world problems.

The paper addresses the problem of developing Hadamard-type inequalities for modified $(h,m)-$convex functions via the Caputo $k-$fractional derivatives. The authors propose a new approach to estimate the fractional derivative of modified $(h,m)-$convex functions through two integral identities involving the nth order derivatives of given functions. The obtained results can have significant applications in various fields of engineering and physics, including the modeling of complex systems governed by fractional differential equations.

The paper provides a detailed overview of the concepts related to the Caputo $k-$fractional derivative and modified $(h,m)-$convex functions. The authors discuss the basic properties of modified $(h,m)-$convex functions and provide several examples of such functions. The paper also presents some important properties of the Caputo $k-$fractional derivative.

The main contribution of the paper is the derivation of several Hadamard-type inequalities for modified $(h,m)-$convex functions via the Caputo $k-$fractional derivative. The authors provide detailed proofs of the obtained results and illustrate their validity through numerical examples. The obtained results are compared with the existing results in the literature, and it is shown that the proposed approach provides sharper bounds.

In conclusion, the paper "Hadamard-type inequalities for modified $(h,m)-$convex functions via Caputo $k-$fractional derivatives" presents an important contribution to the field of fractional calculus. The proposed approach provides a new and efficient way to estimate the fractional derivative of modified $(h,m)-$convex functions. The obtained results have important applications in various fields of engineering and physics, and can contribute to the development of more accurate and efficient models for complex systems governed by fractional differential equations. The paper is well-written and provides a comprehensive overview of the related concepts and results, making it accessible to a broad audience in the field.

5. Future Directions

Here is a detailed list of future directions that could be pursued based on the results and implications of the paper:

Extension of the current results to other fractional derivative operators: While the Caputo k-fractional derivative is widely used, there are several other fractional derivative operators that could be considered. Extending the results of this paper to other fractional derivative operators could lead to new insights and applications in various fields.

Investigation of other types of convexity: In this paper, the modified (h, m)-convexity has been used. However, there are other types of convexity, such as quasi-convexity and pseudo-convexity, that could also be investigated in the context of fractional calculus.

Generalization of the results to vector-valued functions: The results in this paper have been developed for scalar-valued functions. However, many real-world problems involve vector-valued functions. Therefore, it would be interesting to extend the results to vector-valued functions and investigate the properties of such functions.

Applications in optimization: Convex functions have several applications in optimization problems. The results of this paper could be used to develop new optimization techniques that could be applied to real-world problems.

Application to control theory: Fractional calculus has found several applications in control theory. The results of this paper could be used to develop new control strategies for complex systems governed by fractional differential equations.

Investigation of higher-order fractional derivatives: The current results have been developed for the Caputo k-fractional derivative. However, there are several other types of fractional derivatives, such as Riemann-Liouville and Grünwald-Letnikov fractional derivatives, that could be investigated in the context of modified (h, m)-convex functions.

Development of numerical methods: The results of this paper could be used to develop new numerical methods for solving complex systems governed by fractional differential equations.

Investigation of other types of inequalities: While the main focus of this paper has been on Hadamard-type inequalities, there are several other types of inequalities, such as Jensen's inequality and Hölder's inequality, that could be investigated in the context of modified (h, m)-convex functions.

Application to finance: Convexity plays an important role in finance, particularly in the pricing of options. The results of this paper could be applied to develop new pricing models for options that are based on modified (h, m)-convex functions.

Investigation of non-local fractional derivatives: The current results have been developed for local fractional derivatives. However, there are several non-local fractional derivatives, such as the fractional Laplacian, that could be investigated in the context of modified (h, m)-convex functions.

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References

[1] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Symposium. Pure. Math, Amer. Math. Soc., 3(1961) 7-38.
[2] M. F. Atiyah and I. M. Singer, The index of elliptic operators, Ann. Math., 87(1968) 484- 530,564-604, 93(1971) 119-138.
[3] R, Bott, The stable homotopy of the classical groups, Ann. Math., 70(1959) 313-337.
[4] A. Borel and J. P. Serre, Le theoreme de Riemann-Roch (d'apres Grothendieck), Bull. Soc. Math. France., 86(1958) 97-136.
[5] A. Connes, Noncommutative Geometry, Academic Press (1994).
[6] M. Caputo, Linear models of dissipation whose Q is almostfrequency independent-II, Geophys. J. Int., 13(5) (1967) 529-539.
[7] G. Farid, A. Javed and A. U. Rehman, On Hadamard inequalities for $n-$times differentiable functions which are relative convex via the caputo $k-$fractional derivatives, Nonlinear Anal. Forum., 22(2) (2017), 17-28.
[8] G. Farid, N. Latif. M. Anwar, A. Imran, M. Ozair and M. Nawaz, On applications of Caputo k-fractional derivatives, Adv. Differ. Equ., 2019, 2019:439.
[9] G. Farid, A. U. Rehman and M. Zahra, On Hadamard inequalities for k-fractional integrals, Nonlinear Funct. Anal. Appl., 21(3) (2016), 463-478.
[10] W. Gajda, On K and classical conjectures in the arithmetic of cyclotomic fields, Contemporary Math. 346. American Math. Society(2005).
[11] R. Gorenflo and F. Mainardi, Fractional calculus: Integral and differential equations of fractional order, Springer Verlag, Wien, (1997).
[12] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam, (2006).
[13] M. Lazarevi ´c, Advanced topics on applications of fractional calculus on control problems, System stability and modeling, WSEAS Press, Belgrade, Serbia, (2012).
[14] A. S. Merkurjev and A. A. Suslin, K-cohomology of Severi-Brauer varieties and the norm residue homomorphism, Math. USSR Izv., 21 (1983), 307-440.
[15] P. O. Mohammed and T. Abdeljawad, Modification of certain fractional integral inequalities for convex functions, Adv. Differ. Equ. 2020, 2020:69
[16] P. O. Mohammed and T. Abdeljawad, Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel, Adv. Differ. Equ. 2020, 2020:363.
[17] P. O. Mohammed and I. Brevik, A New Version of the HermiteHadamard Inequality for RiemannLiouville Fractional Integrals, Symmetry 12, 610 (2020).
[18] S. Mubeen and G. M. Habibullah, k-Fractional integrals and applications, Int. J. Contemp. Math. Sci., 7 (2012), 89-94.
[19] K. Oldham and J. Spanier, The fractional calculus theory and applications of differentiation and integration to arbitrary order, Academic Press, New York-London (1974).

[20] I. Podlubni, Fractional differential equations, Academic press, San Diego, (1999).
[21] A. Waheed, A. U. Rehman, M. I. Qureshi, F. A. Shah, K. A Khan and G. Farid On Caputo fractional derivatives and associated inequalities, IEEE Access 7,32137-32145 (2019).
[22] M. Z. sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related functional inequalities, Math. Comput. Model, 57(2013), 2403-2407.
[23] G. Toader, Some generalization of convexity, pp. 329-338, Proc. Colloqu. Approx. Optim., Cluj-Napoca, Romania, 1984.
[24] L. N. Mishra, Q. U. Ain, G. Farid and A. U. Rehman, k-fractional integral inequalities for (h,m)-convex functions via the Caputo k-fractional derivatives, Korean. J. Math., 27(2)(2019), 357-374.
[25] A. W. Roberts and D. E. Varberg, Convex functions, Academic press New York and London (1973).
[26] S. Varosanec, On $h-$convexity. J. Math. Anal. Appl. 2007, 326(1), 303-311.