Journal of Corrosion and Materials
www.corromat.com

J. Corros. Mater. **48** (2023) 19–28
https://doi.org/10.61336/jcm2023-2

Higher Weights Geometry of Second Order Tangent Groups and Grassmannian Affine Configuration Chain Complexes

Shanwen Ke[1,*], Hui Wang[1]

[1] *Department of Public Subject Teaching, Shangqiu Medical College, Shangqiu 476299, Henan, China*

(Received: 2 March 2023. Received in revised form: 3 September 2023. Accepted: 9 September 2023. Published online: 11 September 2023.)

Abstract

In this paper, geometry of second order tangent group complex and Grassmannian configuration chain complex is discussed in detail. Initially geometry for special case weight $n = 4$, is introduced, lastly this concept is extended for higher weight $n = 5$.

Keywords: Second order Tangent groups, Configuration complexes, Polylogarithms groups, Truncated polynomial ring.

1. Introduction

Suslin [1] connected free abelian groups by differential boundary maps to define his famous chain complex called Grassmannian configuration chain complex. Configuration chain complex have many applications in algebraic K-theory and geometry. Leibniz introduced Polylog function

$$Li_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^p}, \quad z \leq 1,$$

with relation $log(uv) - log(u) - log(v) = 0$. For weight 1 Bloch defined polylog groups $\mathcal{B}_1(F)$, it a quotient of Z-module $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ and its subgroup, generated by the relation $[x] + [y] - [xy]$ [15]. Bloch also introduced dilogarithmic group $\mathcal{B}_2(F)$, it is quotient of $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ and the relation $\sum_{i=0}^{4}(-1)^i r(v_0, ..., \hat{v_i}, ..., v_4)$ called five term relation of cross ratio of four points. Bloch also introduced chain complex for dilogarithmic group called Bloch-Suslin chain complex. Goncharov [6–8] generalized Bloch group for any weight $n \in \mathcal{N}$ to define generalized polylogarithmic chain complex called Goncharov complex.

Goncharov [6] connected Bloch Suslin and configuration chain complexes by morphisms for the geometry of weight 2 and 3. Using derivation Cathelineau [3–5] introduced variant of Goncharov chain complex in two setting, one was infinitesimal while other was tangential. Siddiqui [12] introduced both cross ratio and the Siegels cross ratio properties in tangential form. Siddiqui [12] also defined geometry of configuration and first order tangential chain complexes for weight 2 and weight 3. Khalid et. al [11] extend the work of [12] up to weight 5. Hussain [14] introduced second order tangent group and its chain complexes for weight 2 and 3. Hussain [14] also defined geometry of second order tangent group and configuration chain complexes both for weight 2 and 3.

Here this work extend the work of [14] to introduced higher weights geometry between Grassmannian configuration and 2nd order tangent groups chain complexes. Section 2 provide basic ideas of Grassmannian affine configuration chain complexes, truncated polynomial and its rings, cross ratio and duel numbers cross ratio, classical polylogarithmic groups and its complexes, first and second order Tangent groups and its generalized chain complexes, geometry of second order tangential and affine configuration chain complexes up to weight 3. Section 3 initialy describe the geometry of 2nd order tangent group and configuration complexes for weight 4 and finally this geometry is extended for higher weight $n = 5$. For weight 4 initially the sub complex of Grassmannian affine configuration chain complex and sub complex of second order tangent group complex are connected through some interesting homomorphism. After connecting these two chain complexes we will show that the resultant diagram is commutative. Similarly for weight 5 the work done for weight 4 is extended, first new homomorphism are introduced to define geometry between second order tangent group and Grassmannian affine configuration chain complexes then with the help of these suitable morphisms we will prove that it give us a commutative diagram. Last section is conclusion of the whole research work, it conclude that in the previous research work researchers define geometry for both lower weights and lower order for these chain complexes but here geometry for higher order and higher weights is proposed.

*Corresponding author (ksw0011@126.com)

2. Background & Literature Review

2.1 Grassmanian Configuration Chain Complex

Let $GL_n(F)$ be a general linear group of order n. The group action of $GL_n(F) * V^n = V^n$, where the set V^n is n-dimensional vector space with elements $(v_1, ..., v_n)$ called configurations of V^n. Following morphisms are two types of differential maps defined as

$$d(v_0, ..., v_n) \to \sum_{i}^{n} (-1)^i (v_0, ..., \hat{v}_i, ..., v_n), \tag{1}$$

and

$$p(v_0, \ldots, v_n) \to \sum_{i}^{n} (-1)^i (v_i | v_0, \ldots, \hat{v}_i, \ldots, v_n) \tag{2}$$

Let $G_m(n)$ be a free abelian group generated by m elements in n-dimensional vector space then following is Grassmannian configuration chain complex.

$$\begin{array}{ccccc} G_{m+3}(n+3) & \xrightarrow{d} & G_{m+2}(n+3) & \xrightarrow{d} & G_{m+1}(n+3) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_{m+2}(n+2) & \xrightarrow{d} & G_{m+1}(n+2) & \xrightarrow{d} & G_m(n+2) \end{array} \tag{A}$$

Each square in diagram A is commutative (see [1]).

2.2 Affine Configuration Chain Complex

Let F be a field with characteristic 0 and the k^{th} truncated polynomial ring is denoted by $F[\varepsilon]_k := F[\varepsilon]/\varepsilon^k, k \geq 1$. Let $\mathbb{A}^n_{F[\varepsilon]_n}$ be affine space defined over the truncated polynomial $F[\varepsilon]_n$.
Let us the element $v = (a_1, a_2, a_3, ..., a_n)^t \in \mathbb{A}^n_F \backslash (0, 0, 0, ..., 0)^t$ and $v_\varepsilon = (a_{1,\varepsilon}, ..., a_{n,\varepsilon})^t \in \mathbb{A}^n_F$ also $v_{\varepsilon^n} = (a_{1,\varepsilon^{k-1}}, a_{2,\varepsilon^{k-1}}, a_{3,\varepsilon^{k-1}}, ..., a$ $\mathbb{A}^n_F$ [14]. Let $G_m(\mathbb{A}^n_{F[\varepsilon]_k})$ be a free abelian group, generated by $(v_1^*, ..., v_m^*)$ vectors in n-dimensional affine space $\mathbb{A}^n_{F[\varepsilon]_k}$, where the vector is defined as $v^* = v + v_\varepsilon \varepsilon + ... + v_{\varepsilon^{k-1}} \varepsilon^{k-1}$ [14]. Now let us introduced the following differential morphisms

$$d : G_{m+1}(\mathbb{A}^n_{F[\varepsilon]_k}) \to G_m(\mathbb{A}^n_{F[\varepsilon]_k})$$

and

$$p : G_{m+1}(\mathbb{A}^n_{F[\varepsilon]_k}) \to G_m(\mathbb{A}^{n-1}_{F[\varepsilon]_k}),$$

following is generalized affine configuration chain complex.

$$\begin{array}{ccccc} G_{m+1}(\mathbb{A}^n_{F[\varepsilon]_k}) & \xrightarrow{d} & G_m(\mathbb{A}^n_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-1}(\mathbb{A}^n_{F[\varepsilon]_k}) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_m(\mathbb{A}^{n-1}_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-1}(\mathbb{A}^{n-1}_{F[\varepsilon]_k}) & \xrightarrow{d} & G_{m-2}(\mathbb{A}^{n-1}_{F[\varepsilon]_k}) \end{array} \tag{B}$$

2.3 Cross Ratio

Cross ratio of four points $(v_0, ..., v_3)$ is defined as

$$r(v_0, v_1, v_2, v_3) = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)},$$

$(v_0, ..., v_3) \in A^2_F$. Following important property is called Siegel [13] cross ratio property

$$1 = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)} + \frac{\triangle(v_0, v_1)\triangle(v_2, v_3)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}. \tag{3}$$

2.4 Siegel cross ratio property in Polynomial Ring $F[\varepsilon]_k$

Case 1 : For k=1 and n=2,

$$\triangle(v_1^*, v_2^*) = \triangle(v_1^*, v_2^*)_{\varepsilon^0} = \triangle(v_1, v_2). \tag{4}$$

Case 2 : For k =2 and n=2,

$$\triangle(v_1^*, v_2^*) = \triangle(v_1^*, v_2^*)_{\varepsilon^0} + \triangle(v_1^*, v_2^*)_{\varepsilon^1}\varepsilon, \tag{5}$$

$\triangle(v_1^*, v_2^*)_{\varepsilon^1} = \triangle(v_1, v_{2,\varepsilon}) + \triangle(v_{1,\varepsilon}, v_2)$.

Case 3 : For k=3 and n=2,

$$\triangle(v_1^*, v_2^*) = \triangle(v_1^*, v_2^*)_{\varepsilon^0} + \triangle(v_1^*, v_2^*)_{\varepsilon^1}\varepsilon + \triangle(v_1^*, v_2^*)_{\varepsilon^2}\varepsilon^2, \tag{6}$$

$\triangle(v_1^*, v_2^*)_{\varepsilon^2} = \triangle(v_1, v_{2,\varepsilon^2}) + \triangle(v_{1,\varepsilon}, v_{2,\varepsilon}) + \triangle(v_{1,\varepsilon^2}, v_2)$ [12, 14].
Following are cross ratio properties and relation in $F[\varepsilon]_k$

$$\mathbf{r}(v_0^*, ..., v_3^*) = (r_{\varepsilon^0} + r_{\varepsilon^1}\varepsilon + ... + r_{\varepsilon^{k-1}}\varepsilon^{k-1})(v_0^*, ..., v_3^*). \tag{7}$$

In the above eq.(7)

$$r_{\varepsilon^0}(v_0^*, ..., v_3^*) = r(v_0, ..., v_3) = \frac{\triangle(v_0, v_3)\triangle(v_1, v_2)}{\triangle(v_0, v_2)\triangle(v_1, v_3)}, \tag{8}$$

$$r_{\varepsilon^1}(v_0^*, ..., v_3^*) = \frac{\{\triangle(v_0^*, v_3^*)\triangle(v_1^*, v_2^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} - r(v_0, ..., v_3)\frac{\{\triangle(v_0^*, v_2^*)\triangle(v_1^*, v_3^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)}, \tag{9}$$

$$\begin{aligned} r_{\varepsilon^2}(v_0^*, ..., v_3^*) = {} & \frac{\{\triangle(v_0^*, v_3^*)\triangle(v_1^*, v_2^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} - r(v_0^*, ..., v_3^*)\frac{\{\triangle(v_0^*, v_2^*)\triangle(v_1^*, v_3^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)} \\ & - r(v_0, ..., v_3)\frac{\{\triangle(v_0^*, v_2^*)\triangle(v_1^*, v_3^*)\}_\varepsilon}{\triangle(v_0, v_2)\triangle(v_1, v_3)}, \end{aligned} \tag{10}$$

and up to $r_{\varepsilon^{k-1}}(v_0, ..., v_3)$.

2.5 Polylogarithmic Groups Complexes

Leibniz introduced polylogarithmic function by a series

$$Li_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^p}, \quad z \leq 1.$$

This infinite series is absolutely convergent in a unit disc.
Let us assume a Z-module defined over doubly punctured set denoted by $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$,it is also free abelian group generated by an element [x].

2.5.1 Polylogarithmic Group

The polylog Scissor congruence group $\mathcal{B}(F)$ is a factor group of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$ and its subgroup generated by the relation $[v_1] - [v_2] + \left[\frac{v_2}{v_1}\right] - \left[\frac{1 - v_2^{-1}}{1 - v_1^{-1}}\right] + \left[\frac{1 - v_2}{1 - v_1}\right], v_1 \neq v_2$ and $v_1, v_2 \neq 0, 1$, also called Abel five term relation [6].

2.5.2 Bloch polylogarithmic group for Weight-1

In [15] Bloch defined a subgroup $\mathsf{R}_1(F)$ of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$, generated by the three relation $\{v_1v_2\} - \{v_1\} - \{v_2\}$, $(v_1, v_2 \in F^\times)$. The quotient gruop $\mathcal{B}_1(F) = Z[\mathbf{P}_F^1/\{0, 1, \infty\}]/\Big\langle \mathsf{R}_1(F)\Big\rangle$ is called Bloch classical polylogarithmic group for weight $n = 1$. Bloch also introduced an isomorphism $\delta_1 : \mathcal{B}_1(F) \to F^\times$, where $\delta_1 : [v] \to v$.

2.5.3 Polylog Bloch Group for weight 2 and Bloch Suslin Chain Complex

For weight 2 group, let $\mathsf{R}_2(F) = \sum_{i=0}^{4}(-1)^i r(v_0, ..., \hat{v}_i,\ ..., v_4)$ be a subgroup of $Z[\mathbf{P}_F^1/\{0, 1, \infty\}]$, generated by cross ratio of four points relation.

$\mathcal{B}_2(F) = Z[\mathbf{P}_F^1/\{0, 1, \infty\}]/\langle \mathsf{R}_2(F)\rangle$. Bloch also connected $\mathcal{B}_2(F)$ and $\wedge^2 F^\times$ to form following chain complex called the Bloch Suslin chain complex [15].

$$\mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times\ ,$$

morphism δ is defined as $\delta[v]_2 = v \wedge v$

2.5.4 Goncharov Chain Complex for Weight-3

Goncharov [6] introduced $\mathsf{R}_3(F)$ subgroup of $Z[\mathbf{P}^1_F/\{0,1,\infty\}]/$, generated by the triple cross ratio relation of six points given by,

$$\mathsf{R}_3(F)=\sum_{i=0}^{6}(-1)^i Alt_6\Big[\frac{(v_0,v_1,v_3)(v_1,v_2,v_4)(v_0,v_2,v_5)}{(v_0,v_1,v_4)(v_1,v_2,v_5)(v_0,v_2,v_3)}\Big] \tag{11}$$

Goncharov [6] defined a factor group $\mathcal{B}_3(F)=Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\langle \mathsf{R}_3(F)\rangle$. Goncharov also introduced following chain complex for Bloch group $\mathcal{B}_3(F)$

$$\mathcal{B}_3(F)\xrightarrow{\delta}\mathcal{B}_2(F)\otimes F^{\times}\xrightarrow{\delta}\wedge^3F^{\times}$$

Lemma 2.1. $\delta\circ\delta=0$ [6].

2.5.5 Weight-n

Goncharov [6] generalized Bloch group

$$\mathcal{B}_n(F)=Z[\mathbf{P}^1_F/\{0,1,\infty\}]/\Big\langle \mathsf{R}_n(F)\Big\rangle,\quad \mathsf{R}_n(F)$$

is a subgroup of Z-module $Z[\mathbf{P}^1_F]$ and it is also the kernel of the morphism $\delta_n: Z[\mathbf{P}^1_F/\{0,1,\infty\}]\to\mathcal{B}_{n-1}(F)\otimes F^{\times}$. Following is generalized classical polylog chain complex called Goncharov complex.

$$\mathcal{B}_n(F)\xrightarrow{\delta_n}\mathcal{B}_{n-1}(F)\otimes F^{\times}\xrightarrow{\delta_{n-1}}\mathcal{B}_{n-2}(F)\otimes\wedge^2(F)\xrightarrow{\delta_{n-2}}\dots\xrightarrow{\delta_2}\mathcal{B}_2(F)\otimes\wedge^{n-2}(F)\xrightarrow{\delta_1}\wedge^n(F^{\times}) \tag{12}$$

Lemma 2.2. $\delta_{n-1}\circ\delta_n=0$ [6].

2.6 First Order Tangent Group

Let us assume v and $v^{'}$ are elements of field F and $\Big\langle v;v^{'}\Big]_2=[v+v^{'}\varepsilon]-[v]\in\mathbb{Z}[F[\varepsilon]_2]$. Cathelineau [4] defined first order $T\mathcal{B}_2(F)$. It is a quotient group of Z-module generated by the elements $\Big\langle x;x^{'}\Big]_2\in\mathbb{Z}[F[\varepsilon]_2]$ and the following relation

$$\Big\langle v;v^{'}\Big]-\Big\langle w;w^{'}\Big]+\Big\langle\frac{w}{v};\Big(\frac{w}{v}\Big)^{'}\Big]-\Big\langle\frac{1-w}{1-v};\Big(\frac{1-w}{1-v}\Big)^{'}\Big]+\Big\langle\frac{v(1-w)}{w(1-v)};\Big(\frac{v(1-w)}{w(1-v)}\Big)^{'}\Big],$$

$v\neq w$ and $v,w\neq 0,1$, this relation is also called five term relation [4,14].

2.6.1 Tangent Group Chain Complex for Weight 2

Following is tangential chain complex to Bloch Suslin complex for weight 2

$$T\mathcal{B}_2(F)\xrightarrow{\delta_\varepsilon}F\otimes F^{\times}\oplus\wedge^2F^{\times}$$

where

$$\delta_\varepsilon:\Big\langle v,w\Big]_2=\Big(\frac{w}{v}\otimes(1-v)+\frac{w}{(1-v)}\otimes v\Big)+\Big(\frac{w}{(1-v)}\wedge\frac{w}{v}\Big)$$

[4,12].

2.6.2 Tangent Group Chain Complex for Weight 3

Cathelineau [4] introduced tangent group of order 3. $TB_3(F)=\mathbb{Z}[F[\varepsilon]_2]/ker_3$, where ker_3 is the kernel of the morphism $\delta_\varepsilon=\mathbb{Z}[F[\varepsilon]_2]\to TB_2(F)\otimes F^{\times}\oplus F\otimes\mathcal{B}_2(F)$. Following chain complex is tangential chain complex for weight 3 to Goncharov complex.

$$T\mathcal{B}_3(F)\xrightarrow{\delta_\varepsilon}T\mathcal{B}_2(F)\otimes F^{\times}\oplus F\otimes\mathcal{B}_2(F)\xrightarrow{\delta_\varepsilon}F\otimes\wedge^2F^{\times}\oplus\wedge^3F^{\times}$$

2.6.3 Generalized First Order Tangent Group Chain Complex

Cathelineu [4] introduced generalized tangent group $TB_n(F)$ and then constructed following generalized first order tangent group complex for any weight n

$$T\mathcal{B}_n(F)\xrightarrow{\delta_{n,\varepsilon}}\begin{matrix}T\mathcal{B}_{n-1}(F)\otimes F^{\times}\\ \oplus\\ F\otimes\mathcal{B}_{n-1}(F)\end{matrix}\xrightarrow{\delta_{(n-1),\varepsilon}}\dots\xrightarrow{\delta_{1,\varepsilon}}\begin{matrix}T\mathcal{B}_2(F)\otimes\wedge^{n-2}F^{\times}\\ \oplus\\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^{\times}\end{matrix}\xrightarrow{\delta_\varepsilon}(F\otimes\wedge^{n-1}F^{\times})\oplus(\wedge^nF)$$

$\delta_{(n-1),\varepsilon}\circ\delta_{n,\varepsilon}=0$ [4].

2.7 Second Order Tangent Group

In [14] Hussain defined the second order tangent group $T\mathcal{B}_2^2(F)$ is qoutient group of Z-module generated by elements $\Big\langle v; v^{'}, v^{''}\Big] \in \mathbb{Z}[F[\varepsilon]_3]$,
where $\Big\langle v; v^{'}, v^{''}\Big] = \Big[v + v^{'}\epsilon + v^{''}\epsilon^2\Big] - \Big[v\Big]$ and $v, v^{'}, v^{''} \in F$ and the following relation

$$\begin{aligned}
&\Big\langle v; v^{'}, v^{''}\Big] - \Big\langle w; w^{'}, w^{''}\Big] + \Big\langle \Big(\frac{w}{v}\Big); \Big(\frac{w}{v}\Big)^{'}, \Big(\frac{w}{v}\Big)^{''}\Big] \\
&- \Big\langle \Big(\frac{1-w}{1-v}\Big); \Big(\frac{(1-w)}{(1-v)}\Big)^{'}, \Big(\frac{(1-w)}{(1-v)}\Big)^{''}\Big] \\
&+ \Big\langle \Big(\frac{v(1-w)}{w(1-v)}\Big); \Big(\frac{v(1-w)}{w(1-v)}\Big)^{'}, \Big(\frac{v(1-w)}{w(1-v)}\Big)^{''}\Big], \quad v, w \neq 0, 1, \quad v \neq w. \qquad (13)
\end{aligned}$$

2.7.1 Second Order Tangent Group Chain Complex for Weight 2

Hussain [14] also defined the following chain for second order tangential complex to the Bloch Suslin complex for weight 2

$$T\mathcal{B}_2^2(F) \xrightarrow{\delta_{\varepsilon^2}} F \otimes F^\times \oplus \wedge^2 F^\times ,$$

morphism δ_{ε^2} is defined as

$$\begin{aligned}
\delta_{\varepsilon^2}\Big\langle v; w_1, w_2\Big] &= \Big(\frac{2w_2}{v} - \frac{w_1^2}{v^2}\Big) \otimes (1-a) + \Big(\frac{2w_2}{(1-v)} - \frac{w_1^2}{(1-v)^2}\Big) \otimes v + \\
&\Big(\frac{2w_2}{v} - \frac{w_1^2}{v^2}\Big) \wedge \Big(\frac{2w_2}{(1-v)} - \frac{w_1^2}{(1-v)^2}\Big); \quad v, w_1, w_2 \in F
\end{aligned}$$

2.7.2 Second Order Tangent Group Chain Complex for Weight 3

$$T\mathcal{B}_3^2(F) \xrightarrow{\delta_\varepsilon} T\mathcal{B}_2^2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F) \xrightarrow{\delta_\varepsilon} F \otimes \wedge^2 F^\times \oplus \wedge^3 F^\times$$

2.7.3 Generalized Second Order Tangent Group Chain Complex

Following is generalized second order tangent group chain complex

$$T\mathcal{B}_n^2(F) \xrightarrow{\delta_{n,\varepsilon}} \begin{matrix} T\mathcal{B}_{n-1}^2(F)\otimes F^\times \\ \oplus \\ F\otimes\mathcal{B}_{n-1}(F)\end{matrix} \xrightarrow{\delta_{(n-1),\varepsilon}} \cdots \xrightarrow{\delta_{1,\varepsilon}} \begin{matrix} T\mathcal{B}_2^2(F)\otimes\wedge^{n-2}F^\times \\ \oplus \\ F\otimes\mathcal{B}_2(F)\otimes\wedge^{n-3}F^\times\end{matrix} \xrightarrow{\delta_\varepsilon} (F \otimes \wedge^{n-1}F^\times) \oplus (\wedge^n F)$$

$\delta_{(n-1),\varepsilon} \circ \delta_{n,\varepsilon} = 0$ [14].

2.8 Second order Tangential and Grassmannian Affine Configuration Complexes Geometry up to Weight 3

2.8.1 Weight 2 Geometry

Hussain in [14] introduced the following geometry for weight-2.

$$\begin{array}{ccccc}
G_5(\mathbb{A}^2_{F[\varepsilon]_3}) & \xrightarrow{p} & G_4(\mathbb{A}^2_{F[\varepsilon]_3}) & \xrightarrow{g^2_{1,\varepsilon^2}} & T\mathcal{B}_2^2(F) \\
\downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\
G_4(\mathbb{A}^2_{F[\varepsilon]_3}) & \xrightarrow{p} & G_3(\mathbb{A}^2_{F[\varepsilon]_3}) & \xrightarrow{g^2_{0,\varepsilon^2}} & F \otimes F^\times \oplus \wedge^2 F^\times
\end{array} \qquad (C)$$

Theorem 2.1. $g^2_{0\varepsilon^2} \circ d = \delta_{\varepsilon^2} \circ g^2_{1\varepsilon^2}$ [14].

2.8.2 Weight 3 Geometry

Following geometry is for weight 3 as defined in [14]

$$\begin{array}{ccccc}
G_7(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & G_6(\mathbb{A}^3_{F[\varepsilon]_3}) & & \\
\downarrow p & & \downarrow p & & \\
G_6(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & G_5(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{g^3_{1,\varepsilon^2}} & T\mathcal{B}_2^2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F) \\
\downarrow p & & \downarrow p & & \downarrow \delta_{\varepsilon^2} \\
G_5(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{d} & G_4(\mathbb{A}^3_{F[\varepsilon]_3}) & \xrightarrow{g^3_{0,\varepsilon^2}} & F \otimes \wedge^2 F^\times \oplus \wedge^3 F^\times
\end{array} \qquad (D)$$

Theorem 2.2. $g^3_{0\varepsilon^2} \circ d = \delta_{\varepsilon^2} \circ g^3_{1\varepsilon^2}$ [14].

3. Geometry for Higher Weights

3.1 Second order Tangential and Grassmannian Affine Configuration Complexes Geometry up to Weight 4

For this weight connect sub complexes of second order tangent group and Grassmannian configuration chain complexes. For the geometry of weight 4 take sub complexes of Grassmannian affine configuration and second order tangent group chain complexes. Now connect these two chain complexes by introducing two suitable and interesting homomorphisms $g^4_{0_{\varepsilon^2}}$ and $g^4_{1\varepsilon^2}$. These homomorphisms will help to produce a commutative diagram. Following commutative diagram is produce after connecting these two complexes.

$$\begin{array}{ccccc} G_7(\mathcal{A}^5_{F[\varepsilon]_r}) & \xrightarrow{p} & G_6(\mathcal{A}^4_{F[\varepsilon]_2}) & \xrightarrow{g^4_{1,\varepsilon^2}} & T\mathcal{B}^2_2(F) \otimes \wedge^2 F^\times \oplus F \otimes \mathcal{B}_2(F) \otimes F^\times \\ \downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\ G_6(\mathcal{A}^5_{F[\varepsilon]_r}) & \xrightarrow{p} & G_5(\mathcal{A}^4_{F[\varepsilon]_2}) & \xrightarrow{g^4_{0,\varepsilon^2}} & F \otimes \wedge^3 F^\times \oplus \wedge^4 F^\times \end{array} \tag{E}$$

where,

$$g^4_{0_{\varepsilon^2}}(v_0^*,...,v_4^*) = \sum_{i=0}^{4}(-1)^i\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,...,v_4)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)}\Big)\otimes$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_4)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_4)} +$$
$$\sum_{i=0}^{4}(-1)^{i+1}\bigwedge_{i=0}^{4}\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,...,v_4)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_4^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_4)}) \ (i \ mod \ 5) \tag{14}$$

and

$$g^4_{1\varepsilon^2}(v_0^*,...,v_5^*)$$
$$= \frac{1}{6}\sum_{i\neq j}^{5}(-1)^i\Big(\Big\langle r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5); r_\varepsilon(v_i^*,v_j^*|v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*),$$
$$r_{\varepsilon^2}(v_i^*,v_j^*|v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v^*\Big]^2_2 \otimes \prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\wedge$$
$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{i\neq r\\ i=0}}^{5}\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,...,v_5^*)_{\varepsilon^2}}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)}\otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes$$
$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{j\neq r\\ j=0}}^{5}\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,...,v_5^*)^2_\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)}\otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2 \otimes \prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\Big) \ (mod \ 6). \tag{15}$$

Theorem 3.1. $g^4_{0_{\varepsilon^2}} \circ d = \delta_{\varepsilon^2} \circ g^4_{1\varepsilon^2}$

Proof. let $(v_0,...,v_5) \in G_6(\mathcal{A}^4_{F[\varepsilon]_2})$, apply morphism d

$$d(v_0,...,v_5) = \sum_{j=0}^{5}(-1)^j(v_0,...,\hat{v}_j,...,v_5) \tag{16}$$

now apply $g^4_{0_{\varepsilon^2}}$

$$g^4_{0_{\varepsilon^2}} \circ d$$

$$= \sum_{j=0}^{5}(-1)^j \sum_{i=0}^{5}(-1)^i \Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v^*}_j,...,v_5)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)}\Big)\otimes$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+1},\hat{v}_j,...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},\hat{v}_j,...,v_5)} +$$
$$\sum_{j=0}^{5}(-1)^j \sum_{i=0}^{5}(-1)^{i+1} \bigwedge_{i\neq j}^{5} \Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)}\Big) \tag{17}$$

let take again $(v_0,...,v_5) \in G_6(\mathcal{A}^4_{F[\varepsilon]_2})$, apply morphism $g^4_{1\varepsilon^2}$

$$g^4_{1\varepsilon^2}(v_0^*,...,v_5^*)$$
$$= \frac{1}{6}\sum_{i\neq j}^{5}(-1)^i \Big(\Big\langle r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5); r_\varepsilon(v_i^*,v_j^*|v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*),$$
$$r_{\varepsilon^2}(v_i^*,v_j^*|v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v^*\Big]_2^2 \otimes \prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\wedge$$
$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{i\neq r\\ i=0}}^{5}\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,...,v_5^*)_{\varepsilon^2}}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)}\otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2\otimes$$
$$\prod_{j\neq r}^{5}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5) + \sum_{\substack{j\neq r\\ j=0}}^{5}\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,...,v_5^*)^2_\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,...,v_5)}\otimes$$
$$[r(v_i,v_j|v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)]_2 \otimes \prod_{i\neq r}^{5}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,...,v_5)\Big), \tag{18}$$

now apply morphism δ_{ε^2} then using wedge, tensor and Siegel cross ratio properties [13], we get

$$\delta_{\varepsilon^2}\circ g^4_{1\varepsilon^2}$$
$$= \sum_{j=0}^{5}(-1)^j \sum_{i=0}^{5}(-1)^i \Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v^*}_j,...,v_5)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)}\Big)\otimes$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+1},\hat{v}_j,...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},\hat{v}_j,...,v_5)} +$$
$$\sum_{j=0}^{5}(-1)^j \sum_{i=0}^{5}(-1)^{i+1} \bigwedge_{i\neq j}^{5} \Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_5^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_5)}\Big) \tag{19}$$

So from Eq.(17) and Eq.(19), $g^4_{0_{\varepsilon^2}}\circ d = \delta_{\varepsilon^2}\circ g^4_{1\varepsilon^2}$.

3.1.1 Weight 5 Geometry

For the geometry of weight 5 extend the geometry of weight 4. Connect the sub complex of second order tangent complex for weight 5 with sub complex of Grassmannian affine configuration chain complex by two new homomorphisms g^5_{0,ε^2} and g^5_{1,ε^2}. Following is resultant commutative diagram for weight 5 after connecting these famous chain complexes.

$$\begin{array}{ccccc} G_8(\mathcal{A}^6_{F[\varepsilon]_2}) & \xrightarrow{p} & G_7(\mathcal{A}^5_{F[\varepsilon]_2}) & \xrightarrow{g^5_{1,\varepsilon^2}} & T\mathcal{B}_2^2(F)\otimes\wedge^3F^\times \oplus F\otimes\mathcal{B}_2(F)\otimes\wedge^2F^\times \\ \downarrow d & & \downarrow d & & \downarrow \delta_{\varepsilon^2} \\ G_7(\mathcal{A}^6_{F[\varepsilon]_2}) & \xrightarrow{p} & G_6(\mathcal{A}^5_{F[\varepsilon]_2}) & \xrightarrow{g^5_{0,\varepsilon^2}} & F\otimes\wedge^4F^\times \oplus \wedge^5F^\times \end{array} \tag{F}$$

where,

$$g^5_{0,\varepsilon^2}(v_0^*,...,v_5^*) = \sum_{i=j+1}^{5}(-1)^{i+1}\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,...,v_5)} - \frac{\triangle(v_0^*,...,\hat{v^*}_i,...,v_5^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,...,v_5)}\Big)\otimes$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+1},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+2},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)} \wedge \frac{\triangle(v_0,...,\hat{v}_{i+3},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},...,v_5)}{\triangle(v_0,...,\hat{v}_{i+5},...,v_5)}+$$

$$\sum_{j=0}^{5}(-1)^{j+1}\bigwedge_{\substack{j\neq i\\ j=0}}^{5}\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_j,...,v_5^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_j,...,v_5)}-\frac{\triangle(v_0^*,...,\hat{v^*}_j,...,v_5^*)\varepsilon}{\triangle(v_0,...,\hat{v}_j,...,v_5)}\Big)\ (i\ mod\ 6) \tag{20}$$

and

$$\begin{aligned}g^5_{1\varepsilon^2}(v_0^*,...,v_6^*)=&\frac{1}{10}\sum_{i\neq j}^{6}(-1)^i\Big(\Big\langle r(v_i,v_j,v_j|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6);r_\varepsilon(v_i^*,v_j^*,v_k^*|v_0^*,...,\\&\hat{v^*}_i,\hat{v^*}_j,\hat{v^*}_k,...,v_6^*)\Big]_2^2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\\&\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)\\&+\sum_{\substack{i\neq r\neq s\\ i=0}}^{6}\Big(\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)}\Big)\otimes\\&[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\\&\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)\\&+\sum_{\substack{j\neq r\neq s\\ j=0}}^{6}\Big(\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)}\Big)\otimes\\&[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\\&\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\\&+\sum_{\substack{k\neq r\neq s\\ k=0}}^{6}\Big(\frac{\triangle(v_0^*,...,\hat{v^*}_k,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_k,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)}\Big)\otimes\\&[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\\&\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\ (mod\ 6).\end{aligned} \tag{21}$$

Theorem 3.2. $g^5_{0_{\varepsilon^2}}\circ d=\delta_{\varepsilon^2}\circ g^5_{1\varepsilon^2}$

Proof. let $(v_0,...,v_6)$ are points of $G_7(\mathcal{A}^5_{F[\varepsilon]_2})$ apply morphism d

$$d(v_0,...,v_6)=\sum_{j=0}^{6}(-1)^j(v_0,...\hat{v}_j,...,v_6) \tag{22}$$

now apply g^5_{0,ε^2}

$$\begin{aligned}g^5_{0,\varepsilon^2}\circ d=&\sum_{j=0}^{6}(-1)^j\sum_{i=j+1}^{6}(-1)^i\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v}_i,\hat{v}_j,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}\Big)\otimes\\&\frac{\triangle(v_0,...,\hat{v}_{i+1},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},\hat{v}_j,...,v_6)}\wedge\\&\frac{\triangle(v_0,...,\hat{v}_{i+4},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},\hat{v}_j,...,v_6)}+\\&\sum_{j=0}^{6}(-1)^j\sum_{i=j+1}^{6}(-1)^i\bigwedge_{j\neq i}^{6}\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}\Big)\end{aligned} \tag{23}$$

Let take again $(v_0,...,v_6)\in G_7(\mathcal{A}^5_{F[\varepsilon]_2})$, apply map $g^5_{1\varepsilon^2}$

$$g^5_{1\varepsilon^2}(v_0^*,...,v_6^*)=\frac{1}{10}\sum_{i\neq j}^{6}(-1)^i\Big(\Big\langle r(v_i,v_j,v_j|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6);r_\varepsilon(v_i^*,v_j^*,v_k^*|v_0^*,...,$$
$$\hat{v^*}_i,\hat{v^*}_j,\hat{v^*}_k,...,v_6^*)\Big]_2^2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge$$
$$\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)$$
$$+\sum_{\substack{i\neq r\neq s\\ i=0}}^{6}\Big(\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)}\Big)\otimes$$
$$[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\wedge$$
$$\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)$$
$$+\sum_{\substack{j\neq r\neq s\\ j=0}}^{6}\Big(\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_j,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)}\Big)\otimes$$
$$[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,$$
$$\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{k\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)\wedge\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)$$
$$+\sum_{\substack{k\neq r\neq s\\ k=0}}^{6}\Big(\frac{\triangle(v_0^*,...,\hat{v^*}_k,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_k,\hat{v^*}_r,\hat{v^*}_s,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_k,\hat{v}_r,\hat{v}_s,...,v_6)}\Big)\otimes$$
$$[r(v_i,v_j,v_k|v_0,...,\hat{v}_i,\hat{v}_j,\hat{v}_k,...,v_6)]_2\otimes\prod_{i\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_i,\hat{v}_r,\hat{v}_s,...,v_6)\wedge$$
$$\prod_{j\neq r\neq s}^{6}\triangle(v_0,...,\hat{v}_j,\hat{v}_r,\hat{v}_s,...,v_6)\tag{24}$$

now apply morphism δ_{ε^2} then using wedge, tensor and Seigal cross ratio properties [13], we get

$$\delta_{\varepsilon^2}\circ g^5_{1\varepsilon^2}=\sum_{j=0}^{6}(-1)^j\sum_{i=j+1}^{6}(-1)^i\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v}_i,\hat{v}_j,...,v_6^*)^2\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}\Big)\otimes$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+1},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+2},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_6)}\wedge\frac{\triangle(v_0,...,\hat{v}_{i+3},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+4},\hat{v}_j,...,v_6)}\wedge$$
$$\frac{\triangle(v_0,...,\hat{v}_{i+4},\hat{v}_j,...,v_6)}{\triangle(v_0,...,\hat{v}_{i+5},\hat{v}_j,...,v_6)}+$$
$$\sum_{j=0}^{6}(-1)^j\sum_{i=j+1}^{6}(-1)^i\bigwedge_{j\neq i}^{6}\Big(2\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_6^*)\varepsilon^2}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}-\frac{\triangle(v_0^*,...,\hat{v^*}_i,\hat{v^*}_j,...,v_6^*)\varepsilon}{\triangle(v_0,...,\hat{v}_i,\hat{v}_j,...,v_6)}\Big),\tag{25}$$

from Eq.(23) and Eq.(25), $g^5_{0_{\varepsilon^2}}\circ d=\delta_{\varepsilon^2}\circ g^5_{1\varepsilon^2}$.

4. Conclusion

In this research work geometry of second order tangent and Grassmannian affine configuration chain complexes for higher weights 4 and 5 is proposed to produce commutative diagrams. Homomorphisms to connect Grassmannian affine configuration and second order tangent groups complexes for both weight 4 and 5 are interesting and helpful to introduce generalized geometry for these two chain complexes. For the first time higher weight geometry of second order tangent group is proposed as other researcher introduced geometry for lower weights two and three.

References

[1] Suslin, A.A., *Homology of GL_n, characteristic classes and Milnor's K-theory*, Proceedings of the Steklov Institute of Mathematics 1985, Issue 3, pp. 207-226; Lecture Notes in Math., vol. 1046, Springer-Verlag, New York, 1989, pp. 35-375.

[2] Bloch,S and Esnault, H.,The additive dilogarithm, Kazuya Kato's fiftieth birthday, Doc. Math.(2003), Extra Vol. 131-155.

[3] Cathelineau,J-L.,Infinitesimal Polylogarithms, multilicative presentation of Kähler Differential and Gonchrove Complexes, talk at the workshop on polylogarthms, Essen, May 1-4 (1997)

[4] Cathelineau, J-L., *The tangent complex to the Bloch-Suslin complex*, **Bull**. Soc.Math.France **135**(2007)565-597.

[5] Cathelineau, J-L., *Remarques sur les Différentielles des Polylogarithmes Uniformes*, Ann. Inst. Fourier, Grenoble **46**, (1996)1327-1347.

[6] Goncharov,A. B., Geometry of configuration, Polylogarithms and Motivic Cohomology, Adv.Math.,144(1995) 197-31

[7] Goncharov AB, Zhao J. Grassmannian trilogarithm, Compositio Mathematica 2001; 127: 83-108.

[8] Goncharov AB. Euclidean scissor congruence groups and mixed Tate motives over dual numbers, Mathematical Research Letters 2004; 11: 771-784.

[9] J. L. Dupont, *Scissors congruences, group homology and characteristic classes ,Nankai Tracts in Mathematics*, vol. 1,World Scintefic Publishing Co., Inc ., River Edge, NJ, (2001).

[10] Matsumura, H., *Commutative Ring Theory*, Cambridge University Press, 1986.

[11] M. Khalid, M. Sultana, J.Khan and A. Iqbal, *Geometry of Tangential and configuration chain complexes for higher weights*, Punjab Univ. J. Math., **52**(2) (2020) 29-55.

[12] R. Siddiqui (2010)Configuration Comlexes and Tangential and Infinitesimal version of poly logarithmic Complexes.

[13] Siegel , C.L., *Approximation algebraischer Zahlen*,Mathem.Ze/tschr.10(1921),173-213.

[14] S.Hussain, *Grassmannian Complex and Second Order Tangent Complex*, Pujm.maths. 48(2)(2016),91-111.

[15] S. Bloch, *Applications of the dilogarithm function in algebraic K-theory and algebraic geometry*, Proc. int. Symp. on Algebraic Geometry (Kyoto Univ., Kyoto, Japan), pp. 103?114 1978.