**Journal of Corrosion and Materials**
www.corromat.com

*J. Corros. Mater.* **48** (2023) 29–34
https://doi.org/10.61336/jcm2023-3

# Cyclic-coverings of Line graph $L(S_n)$ and its Disjoint Union

Xiaojun Ren[1,*], Shaochun Chen[1]

[1] *Blockchain Laboratory of Agricultural Vegetables, Weifang University of Science and Technology, Shouguang, 262700, China*

(Received: 12 March 2023. Received in revised form: 03 September 2023. Accepted: 19 September 2023. Published online: 21 September 2023.)

**Abstract**

A graph is a particular representation of a static network and labeling of a graph can be think of as automatic routing of data in a network topology. A visual representation of data in the form of a graph help us to look deep insight. The data science (e.g., Python Package: a computer programming language) uses graph theory concepts to study and analyze the networks. Analyzing these network is equivalent of finding a set of edges $E'$ for a graph $G$ such that every vertex of $G$ is incident with at least one edge in $E'$. Then $E'$ is called an *edge-covering* of $G$. A spanning tree of a connected graph is an example of edge-covering. A finite simple graph $G$ is an $(a_d, d)$-$H$-antimagic if the following three conditions are satisfied: $G$ has an $H$-covering ($H$ a subgraph of $G$), there exists a bijection $\alpha : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G) \cup E(G)|\}$ and the $H$-weights constitute an arithmetic progression with common difference $d$. The above said labeling is called super if $\alpha(V) = \{1, 2, \ldots, |V|\}$. In this research article, we focused on studying super $C_3$-antimagic labeling of line graph of a sun-let graph for several differences and $C_3$-supermagic labeling of its disjoint union.

**Keywords:** Network representation, $C_3$-coverings, star graph, sun-let graph, line graph, disjoint union of graphs.

## 1. Introduction

A graph is a particular representation of a static network and labeling of a graph can be think of as automatic routing of data in a network topology [17]. Vertices (or nodes) of a graph represents computers while edges (or curves) represents connection between computers of a network. Antimagic labeling of a graph helps routers to detect and distinguish between different computers on the same network. Network visualization using graph enables us to analyze the network structure. Mobile Adhoc Networks (MANETS) [7] issues can be resolved using graph labeling. The solutions of MANETS in Graph theory are found by graph spanners and the proximity of graph. All these problems can be addressed by finding an edge-coverings of a graph and then assigning them different labels (weights).

Let $G = (V_G, E_G)$ be a connected, finite and simple graph. An *edge-covering* of $G$ is a family $\{G'_1, G'_2, \ldots, G'_q\} \subset G$ such that for all $e \in E_G$, $e \in G'_l$, for some $l$, $l = 1, 2, \ldots, q$. If $G'_l \cong G'$, $\forall l$, then $G$ has an $G'$-*covering*. Graph $G$ with $G'$-covering is an $(a_d, d)$-$G'$-antimagic if there exists a bijection $\alpha : V_G \cup E_G \to \{1, 2, \ldots, |V_G| + |E_G|\}$ such that for all subgraphs of $G$ isomorphic to $G'$

$$\begin{aligned} wt_\alpha(G') &= \{\sum(\alpha(v_{G'}) + \alpha(E_{G'}))\} \\ &= \{a_d, a_d + d, \ldots, a_d + (q-1)d\} \end{aligned}$$

where $a_d > 0$ and $d \geq 0$ are positive integers. For $\alpha(V_G) = \{1, 2, 3, \ldots, |V_G|\}$, the labeling $\alpha$ becomes *super* $(a_d, d)$-$G'$-*antimagic* and it would be $G'$-*supermagic* for $d = 0$. Gutiérrez and Lladó in [3] defined a *super G-magic labeling.* The obtained results are about star graphs, complete bipartite graphs, wheels, prisms and banana tree graphs. The $C_n$-supermagic labeling of different families of graphs can be found in [9]. Examples of $G'$-supermagic graphs with different choices of $G'$ are given by Jeyanthi and Selvagopal in [6]. Inayah *et al.* [4] gave idea of an $(a_d, d)$-$G'$-antimagic labeling. The results about $(a_d, d)$-$G'$-antimagic labeling of shackles of a connected graph $G$ are in [4, 5].

In recent years, Baca *et al.* [2] discussed the tree-antimagicness of disconnected graphs. Authors in [10] discussed the super $(a, 1)$-tree-antimagicness of sun graphs. The antimagic behaviour of generalized sun graph and its subdivision are discussed in [18]. The results about $b$-chromatic number of a sun-let related graphs and a wheel graph are discussed in [20]. Author in [21] discussed the problem of deciding whether an arbitrary graph contains a sun is NP-complete. Mahavir [22] obtained results about the printing cycle for embedding the sun graph in a single page. In the same paper, he gave a linear time algorithm for such an embedding. Graceful labelings for sun graph, extension sun, double fan and leg graph with star are discussed in [23].

In the present paper, we proved the results about super $(a_d, d)$-$C_3$-antimagic labeling of line graph $L(S_n)$ for differences $d \in \{0, 1, \ldots, 7, 8, 9, 10, 11, 13\}$ and $C_3$-supermagic labeling for disjoint union of $L(S_n)$, where $S_n$ is a star graph on $n$ vertices.

*Corresponding author (renxiaojun_05@yeah.net)

## 2. $C_3$-antimagic labeling of line graph of Sun-let graph

The *sun-let* graph ($n$-sun let) $S_n,\ n \geq 3$, is obtained from a cycle $C_n$ by attaching $n$ pendant edges to a cycle $C_n$.
[19] The *line graph* $L(G)$ of a graph $G$ is graph with $V(L(G)) = E(G)$ and $E(L(G)) = \{ee' : e,\ e'$are incident edges in $G\}$. Let $S_n, n \geq 3$ be a sun-let graph with vertex set $V(S_n) = \{x_t, y_t : 1 \leq t \leq n\}$ and edge set $E(S_n) = \{x_t x_{t+1} : 1 \leq t \leq n-1\} \cup \{x_t y_t : 1 \leq t \leq n\}$. The line graph $L(S_n)$ has vertex set: $V(L(S_n)) = \{x'_t, y'_t : 1 \leq t \leq n\}$ and edge set: $E(L(S_n)) = \{x'_t, x'_{t+1}\} \cup \{y'_t x'_t, y'_t x'_{t-1}\}$, where indices $t$ are taken modulo $n$. Clearly Line graph $L(S_n)$ of a sun-let graph has $C_3$ coverings of the form $\{x'_t, y'_t, x'_{t+1}\}$, where indices $t$ are taken modulo $n$. Figure 1 depicts line graph $L(S_n)$ of a sun-let graph. Under a total labeling $\alpha$, the $C_3^{(j)}$-weights, for indices $j = 1, 2, \ldots, n$ are:

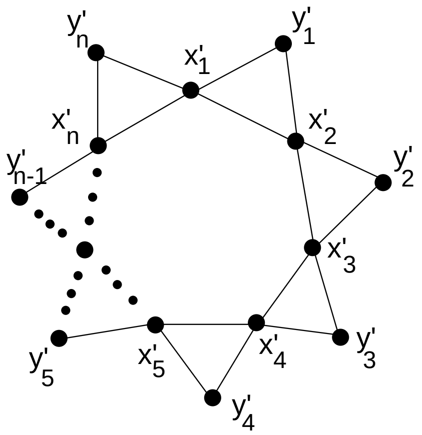


Figure 1: Line graph $L(S_n)$ of a sun-let graph $S_n$

$$\begin{aligned} wt_\alpha(C_3^{(j)}) &= \sum_{u \in V(C_3^{(j)})} \alpha(u) + \sum_{e \in E(C_3^{(j)})} \alpha(e). \\ &= \left(\alpha(x'_j) + \alpha(x'_{j+1}) + \alpha(y'_j)\right) + \\ &+ \left(\alpha(x'_j)\alpha(x'_{j+1}) + \alpha(x'_j)\alpha(y'_j) + \right. \\ &\left. + \alpha(x'_{j+1})\alpha(y'_j)\right) \end{aligned} \tag{1}$$

**Theorem 2.1.** *The line graph $L(S_n)$ of sun-let graph $S_n$ admits a super $(a_d, d)$-$C_3$-antimagic labeling for $a_d > 0,\ n \geq 3$ positive integers and $d \in \{0, 1, \ldots, 7, 10, 11\}$.*

*Proof.* The total labeling $\alpha_d,\ d = 0, 1, \ldots, 7, 10, 11$ for line graph $L(S_n)$ is defined as:

$$\alpha_d(x'_i) = \begin{cases} i & d = 0, 2, 4, 6, 10 \\ 2i & d = 1, 3, 5, 7, 11 \end{cases}$$

$$\alpha_d(y'_i) = \begin{cases} 2n + 1 - i & d = 0, 2, 4, 6, 10 \\ 2n + 1 - 2i & d = 1, 3, 5, 7, 11 \end{cases}$$

$$\alpha_d(x'_i x'_{i+1}) = \begin{cases} 2n + 1 + i & d = 0, 1, 2, \ldots, 7 \\ 2n + 1 + 3i & d = 10, 11 \end{cases}$$

$$\alpha_d(x'_i y'_i) = \begin{cases} 5n - i & d = 0, 1, 2, 3 \\ 4n + 1 + i & d = 4, 5 \\ 3n + 2 + 2i & d = 6, 7 \\ 2n + 3 + 3i & d = 10, 11 \end{cases}$$

$$\alpha_d(y'_i x'_{i+1}) = \begin{cases} 4n-i & d=0,1 \\ 3n+1+i & d=2,3,4,5 \\ 3n+1+2i & d=6,7 \\ 2n+2+3i & d=10,11 \end{cases}$$

where indices $i$ are taken modulo $n$.
Using (1), the the $C_3^{(j)}$-weights are:

$$\begin{aligned} wt_{\alpha_0}(C_3^{(j)}) &= (2(n+1)+i)+(11n+1-i) = 13n+3 \\ wt_{\alpha_1}(C_3^{(j)}) &= (2n+3+2i)+(11n+1-i) = 13n+4+i \\ wt_{\alpha_2}(C_3^{(j)}) &= (2n+2+i)+(10n+2+i) = 12n+4+2i \\ wt_{\alpha_3}(C_3^{(j)}) &= (2n+3+2i)+(10n+2+i) = 12n+5+3i \\ wt_{\alpha_4}(C_3^{(j)}) &= (2n+2+i)+(9n+3+3i) = 11n+5+4i \\ wt_{\alpha_5}(C_3^{(j)}) &= (2n+3+2i)+(9n+3+3i) = 11n+6+5i \\ wt_{\alpha_6}(C_3^{(j)}) &= (2n+2+i)+(8n+4+5i) = 10n+6(i+1) \\ wt_{\alpha_7}(C_3^{(j)}) &= (2n+3+2i)+(8n+4+5i) = 10n+7(i+1) \\ wt_{\alpha_{10}}(C_3^{(j)}) &= (2n+2+i)+(6n+6+9i) = 8(n+1)+10i \\ wt_{\alpha_{11}}(C_3^{(j)}) &= (2n+3+2i)+(6n+6+9i) = 8n+9+11i \end{aligned} \quad (2)$$

Clearly total labeling $\alpha_d,\ d=0,1,\ldots,7,10,11$ is super since vertices are labelled with integers $\{1,2,\ldots,2n\}$. Equation (2) shows that line graph $L(S_n)$ admits super $(a_d,d)$-$C_3$-antimagic labeling for $d\in\{0,1,\ldots,7,10,11\}$.
This completes the proof of theorem. □

**Theorem 2.2.** *The line graph $L(S_n)$ of Sun-let graph $S_n$ admits a super $(a_d,d)$-$C_3$-antimagic labeling for $(n\geq 3)$ odd, $a_d>0$ positive integers and $d\in\{8,9,13\}$.*

*Proof.* The total labeling $\alpha_d,\ d=8,9$ for line graph $L(S_n)$ is defined as:

$$\alpha_d(x'_i) = \begin{cases} \frac{n+i+1}{2} & i\equiv 0 \pmod 2, d=8,9 \\ n+i+1 & i\equiv 0 \pmod 2, d=13 \\ \frac{i+1}{2} & i\equiv 1 \pmod 2, d=8,9 \\ i+1 & i\equiv 1 \pmod 2, d=13 \end{cases} \quad (3)$$

$$\alpha_d(y'_i) = \begin{cases} 2n+1-\frac{n+1+i}{2} & i\equiv 0 \pmod 2, d=8 \\ 2n+1-\frac{i+1}{2} & i\equiv 1 \pmod 2, d=8 \\ 2n-i & d=9 \\ 2i+1 & d=13 \end{cases} \quad (4)$$

Clearly from equations (3) and (4) for any $C_3^{(j)}$, we have

$$wt_{\alpha_d}(V(C_3^{(j)})) = \begin{cases} 2n+2+\frac{i}{2} & i\equiv 0 \pmod 2, d=8 \\ \frac{5n+4+i}{2} & i\equiv 0 \pmod 2, d=8 \\ \frac{5n+3}{2} & d=9 \\ n+4(i+1) & d=13 \end{cases} \quad (5)$$

where indices $i$ are taken modulo $n$.
For $j=1,2,\ldots,n$, we have

$$\alpha_8(E(C_3^{(j)})) = \begin{cases} \bigcup_{k=0}^{2}\{5n-k-3(\frac{j}{2})\} & i\equiv 0 \pmod 2 \\ \bigcup_{k=0}^{2}\{5n-k-3(\frac{n+j}{2})\} & i\equiv 1 \pmod 2 \end{cases} \quad (6)$$

$$wt_{\alpha_8}(E(C_3^{(j)})) = \begin{cases} 17n-1-4i & i\equiv 0 \pmod 2 \\ 13n-1-4i & i\equiv 1 \pmod 2 \end{cases} \quad (7)$$

and for $d\in\{9,13\}$

$$\alpha_d(E(C_3^{(j)})) = \bigcup_{k=0}^{2}\{2(n-1)+3j+k\} \quad (8)$$

$$wt_{\alpha_d}(E(C_3^{(j)})) = 3(2n-1) + 9j \tag{9}$$

Using equations (1), (5),(6),(7), the $C_3$-weights under labeling $\alpha_8$ are $\{9n+7, 9n+15, \ldots, 17n-1)\}$ which clearly constitute an arithmetic progression with a common difference $d = 8$ and labeling $\alpha_8$ is super.
Similarly using equations (1), (5), (8), (9), the $C_3$-weights under labeling $\alpha_d, d \in \{9, 13\}$ are:

$$wt_{\alpha_9}(C_3^{(j)}) = \frac{5n+3}{2} + 6(n+1) + 9i \tag{10}$$

$$wt_{\alpha_13}(C_3^{(j)}) = 7n + 10 + 13i \tag{11}$$

Equation (10) constitute an arithmetic progression with $a = \frac{17n+15}{2}$ and a common difference $d = 9$. Equation (11) constitute an arithmetic progression with $a = 7n + 10$ and a common difference $d = 13$.
This completes the proof of our theorem. □

## 3. Cycle antimagic labeling of Disjoint union of $L(S_n)$

**Theorem 3.1.** *Let line graph $L(S_n)$ of sun-let graph $S_n$, $n \geq 3$ admits a $C_3$-supermagic labeling. Then the disjoint union of arbitrary number of copies of $L(S_n)$, i.e. $sL(S_n)$ also admits a $C_3$-supermagic labeling for $s \geq 1$ a positive integer.*

*Proof.* In the proof of our theorem, $\Gamma$ denotes $L(S_n)$, $u_k, k = 1, 2, \ldots, s$, denotes a vertex or an edge in the $k^{th}$ copy of the line graph $\Gamma = L(S_n)$ of a sun-let graph, denoted by $\Gamma^{(k)} := L(S_n)(k)$, corresponding to $u$ in $\Gamma = L(S_n)$, i.e., $u \in V(L(S_n)) \cup E(L(S_n))$. In the same way $C_3^{(r)}(k), k = 1, 2, \ldots, s,\ r = 1, 2, \ldots, n$, be the subgraph in the $k^{th}$ copy of $L(S_n)$ corresponding to the subgraph $C_3^{(r)}$ in $L(S_n)$.

The total labeling $\alpha'$ of $s\Gamma$ is defined as:

$$\alpha'(u_k) = \begin{cases} m(\alpha(u) - 1) + k & \text{if } u \in V(\Gamma) \\ m\alpha(u) + 1 - k & \text{if } u \in E(\Gamma) \end{cases}$$

First, we will prove that vertices of $\bigcup_{k=1}^{s} \Gamma^{(k)}$ use integers from 1 up to $ps$ under the labeling $\alpha$ where $p$ is number of vertices in graph $\Gamma$. i.e.,

$$\alpha'(V(\Gamma^{(k)})) = \begin{cases} \{1, s+1, 2s+1, \ldots, & (p-1)s+1\} \\ & k = 1 \\ \{2, s+2, 2s+2, \ldots, & (p-1)s+2\} \\ & k = 2 \\ \quad\ldots & \\ \{j, s+j, 2s+j, \ldots, & (p-1)s+j\} \\ & k = j \\ \quad\ldots & \\ \{s, 2s, 3s, \ldots, ps\} & k = s \end{cases} \tag{12}$$

Secondly, for edges of $\bigcup\limits_{k=1}^{s} \Gamma^{(k)}$ under the labeling $\alpha$ with $|E(\Gamma)| = q$, we have:

$$\alpha'(E(\Gamma^{(k)})) = \begin{cases} \{(p+1)s, (p+2)s, (p+3)s, \ldots\} \\ \cup\{(p+q)s\} \quad k = 1 \\ \{(p+1)s-1, (p+2)s-1, \} \\ \cup\{(p+3)s-1, \ldots, (p+q)s-1\} \\ k = 2 \\ \quad\ldots \\ \{(p+1)s+1-j, (p+2)s+1-j\} \\ \cup\{(p+3)s+1-j, \ldots, (p+q)s+1-j\} \\ k = j \\ \quad\ldots \\ \{ps+1, (p+1)s+1, (p+2)s+1, \ldots\} \\ \cup\{(p+q-1)s+1\} \\ k = s \end{cases} \tag{13}$$

From equations (12) and (13), it is clear the labeling $\alpha'$ is a total labeling since $\alpha'$ is a bijection between the integers $\{1, 2, \ldots, (p+q)s\}$ and the vertices and edges of $\bigcup_{k=1}^{m} \Gamma^{(k)}$.

Under the total labeling $\alpha'$, the $C_3^{(r)}(k)$-weights, $1 \leq k \leq s, 1 \leq r \leq n$ would be:

$$\begin{aligned}
wt_{\alpha'}(C_{(3,k)}^{(r)}) &= \sum_{v \in V(C_3^{(r)}(k))} \alpha'(v) + \sum_{e \in E(C_3^{(r)}(k))} \alpha'(e) \\
&= \sum_{v \in V(C_3^{(r)}(k))} (s(\alpha'(v) - 1) + k) + \\
&+ \sum_{e \in E(C_3^{(r)}(k))} (s\alpha'(e) + 1 - k) \\
&= s \sum_{v \in V(C_3^{(r)}(k))} \alpha'(v) - s|V(C_3^{(r)}(k))| + \\
&+ k|V(C_3^{(r)}(k))| + s \sum_{e \in E(C_3^{(r)}(k))} \alpha'(e) + \\
&+ |E(C_3^{(r)}(k))| - k|E(C_3^{(r)}(k))|
\end{aligned}$$

where $n$ is the number of $C_3$'s in $\Gamma^{(k)}$.

$$\begin{aligned}
wt_{\alpha'}(C_{(3,k)}^{(r)}) = s &\left( \sum_{v \in V(C_3^{(r)}(k))} \alpha'(v) + \sum_{e \in E(C_3^{(r)}(k))} \alpha'(e) \right) - \\
&- s|V(C_3^{(r)}(k))| + |E(C_3^{(r)}(k))| \\
&+ k|V(C_3^{(r)}(k))| - k|E(C_3^{(r)}(k))| \\
&= swt_{\alpha'}(C_3^{(r)}(k)) - s|V(C_3^{(r)}(k))| + \\
&+ |E(C_3^{(r)}(k))| + k|V(C_3^{(r)}(k))| - \\
&- k|E(C_3^{(r)}(k))|.
\end{aligned}$$

As every $C_3^{(r)}(k)$, $k = 1, 2, \ldots, s, r = 1, 2, \ldots, n$, is isomorphic to the cycle $C_3$, it holds

$$\begin{aligned}
|V(C_3^{(r)}(k))| &= |V(C_3)| = 3 \\
|E(C_3^{(r)}(k))| &= |E(C_3)| = 3
\end{aligned}$$

Thus for the $C_3$-weights, we get

$$\begin{aligned}
wt_{\alpha'}(C_3^{(r)}(k)) &= swt_{\alpha'}(C_3^{(r)}) + 3(1 - s) \\
&= \frac{s}{2}(29n + 43) + 3(1 - s) \\
&= \frac{s}{2}(29n + 35) + 3.
\end{aligned}$$

It is easy to see that the set of all $C_3^{(r)}(k)$-weights in $\bigcup_{k=1}^{s} \Gamma^{(k)}$ consists of same integers. Thus the graph $\bigcup_{k=1}^{s} \Gamma$ is a $C_3$-supermagic.

This completes the proof of our theorem. □

## References


[1] Noshad Ali, Muhammad Awais Umar, Afshan Tabassum, Abdul Raheem., Super (a,d)-C3-antimagicness of a Corona Graph., Open J. Math. Sci., Vol. 2 (2018), Issue 1, pp. 371 - 278

[2] M. Bača, Z. Kimăkovă, A. Semaničová-Feňovčíková and M.A. Umar, Tree antimagicness of disconnected graphs, *Mathematical Problems in Engineering*, (2015), DOI:10.1155/2015.

[3] A. Gutiérrez and A. Lladó, Magic coverings, *J. Combin. Math. Combin. Comput.* **55** (2005), 43–56.

[4] N. Inayah, A.N.M. Salman and R. Simanjuntak, On $(a, d)$-$H$-antimagic coverings of graphs, *J. Combin. Math. Combin. Comput.* **71** (2009), 273–281.

[5] N. Inayah, R. Simanjuntak, A.N.M. Salman and K.I.A. Syuhada, On $(a, d)$-$H$-antimagic total labelings for shackles of a connected graph $H$, *Australasian J. Combin.* **57** (2013), 127–138.

[6] P. Jeyanthi and P. Selvagopal, More classes of $H$-supermagic graphs, *Internat. J. Algor. Comput. Math.* **3** (2010), 93–108.

[7] Kenta Ozeki, Tomoki Yamashita, Spanning Trees: A survey, *Graphs and Combinatorics* **27**(1), (2011), 1–26, DOI:10.1007/s00373-010-0973-2.

[8] T.K. Maryati, E.T. Baskoro and A.N.M. Salman, $P_h$-(super)magic labelings of some trees, *J. Combin. Math. Combin. Comput.* **65** (2008), 198–204.

[9] A.A.G. Ngurah, A.N.M. Salman and L. Susilowati, $H$-supermagic labelings of graphs, *Discrete Math.* **310** (2010), 1293–1300.

[10] M.A. Umar, M. Hussain, B.R. Ali & M. Numan, Super-Tree-antimagicness of Sun Graphs, *International Journal of Soft Computing and Engineering (IJSCE)*, Vol 7(6), 2018.

[11] A. Raheem, M. Javaid, M. A. Umar, G. C. Lau, On Super (a,d)-EAT valuation of subdivided caterpillar, *TWMS J. App. Eng. Math.*, Vol 9, N. 4(2019). pp. 693–703.

[12] M. A. Umar, M.A. Javed, M. Hussain and B.R. Ali, super $(a,d)$-$C_4$-antimagicness of Book Graphs, *Open Journal of Mathematical Sciences*, **2018**1, 115–121, DOI:10.30538/oms2018.0021.

[13] Muhammad Awais Umar, Noshad Ali, Afshan Tabassum, Basharat Rehman Ali., Book graphs are cycle antimagic, Open J. Math Sci., Vol. 3 (2019), Issue 1, pp. 184 - 190.

[14] Muhammad Awais Umar, Cycle-antimagic construction of Ladders, *Eng. Appl. Sci. Lett.*, Vol. 2, (2019), No. 2, pp. 43 - 47, DOI:10.30538/psrp-easl2019.0020.

[15] Yi Hu, M. A. Umar, Mustafa Habib, Ce Shi, Ghulam Rasool, Zijiang Zhu, Antimagicness of mCn-Path and Its Disjoint Union,, *IEEE Access*, Vol 7, (2019). pp. 147345–147349, DOI:10.1109/ACCESS.2019.2949043.

[16] Yijun Xiong, Huajun Wang, Mustafa Habib, M. A. Umar, Basharat Rehman Ali, Amalgamations and Cycle-antimagicness, *IEEE Access*, Vol 7, (2019). pp. 147345–147349, DOI:10.1109/ACCESS.2019.2936844.

[17] Zongheng Zhou, Samir Das, Himanshu Gupta, Connected K-Coverage Problem in Sensor Networks, *Proceedings: 13th International Conference on Computer Communications and Networks (IC3N*, IEEE Access, (2014), DOI: 10.1109/ICCCN.2004.1401672.

[18] M.A. Umar, M. Hussain, & B.R. Ali, Antimagic Behavior of SGnp and its Subdivision, *Wireless Communications and Mobile Computing*, submitted.

[19] Diestel, Reinhard. *Graph theory*, 3rd ed., Graduate texts in mathematics, 173 (2005).

[20] Vivin, J. Vernold & M. Vekatachalam, On b-chromatic number of sun let graph and wheel graph families, *Journal of the Egyptian Mathematical Society* 23, no. 2 (2015): 215-218.

[21] Hoang, C. T. On the complexity of finding a sun in a graph, *SIAM Journal on Discrete Mathematics* 23, no. 4 (2010): 2156-2162.

[22] Mahavir, B., Embedding Sun Graphs in a Single page, *The Bulletin of Society for Mathematical Services and Standards*, (2013), Vol. 5, pp 31-36, doi:10.18052/www.scipress.com/BSMaSS.5.31.

[23] Solairaju, A., S. Subbulakshmi, and R. Kokila. Graceful Labelings for Sun Graph, Sun Extension Graph, Double Fan, and Leg Graph with Star, *Global Journal of Pure and Applied Mathematics* 13, no. 5 (2017): 1357-1364.