Corrosion and Materials *Corros. Mater.* **48** (2023) 65–68
www.corromat.com

Antimagic Behavior of SG_n^p and its Subdivision

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(Received: 2 January 2023. Received in revised form: 23 October 2023. Accepted: 2 November 2023. Published online: 5 November 2023.)

Abstract

A finite simple graph G with a subgraph H is called a super (b,d)-H-antimagic: if G has an edge covering by subgraphs $H_1, H_2, \ldots, H_t$ with each $H_i \cong H$, $i = 1, 2, \ldots, t$, a total labeling α such that $wt_\alpha H$, constitutes an arithmetic progression and $\alpha(V(G))$ consists of the smallest possible integers. In this manuscript, we investigated the existence of super $(b,1)$-star-antimagic labeling of Sun graphs SG_n^p.

Keywords: Star graph S_n, Sun graph SG_n^p, super $(b,1)$-S_{p+2}-antimagic.

1. Introduction

If each edge of a graph G belongs to one of the subgraphs $H_1, H_2, \ldots, H_t$ with $H_i \cong H$, H_i, $i = 1, 2, \ldots, t$ then G admits an *H-covering*. A graph G with an H-covering is *(b,d)-H-antimagic* if for a total labeling $\alpha : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$, the H-weights,

$$wt_\alpha(H) = \sum_{x \in V(H)} \alpha(x) + \sum_{y \in E(H)} \alpha(y)$$

constitute an arithmetic progression $\{b, b+d, \ldots, b+(t-1)d\}$, where $b > 0$ and $d \geq 0$ are two integers and t is the number of all subgraphs $H_i \cong H$. Furthermore, α is *super (b,d)-H-antimagic labeling* if $\alpha(V(G)) = \{1, 2, \ldots, |V|\}$.

The (super) H-magic graph was first introduced by Gutiérrez and Lladó in [4]. Proved results are about: star $K_{1,n}$ graphs, complete bipartite graphs $K_{n,m}$, paths P_n, cycles C_n, wheels, windmills, books, prisms, shrubs and banana tree graphs. Lladó and Moragas [9] investigated C_n-(super)magic graphs and proved that wheels, windmills, books and prisms are C_h-magic for some h. Some results on C_n-supermagic labelings of several classes of graphs can be found in [13]. Maryati et al. [10] gave P_h-(super)magic labelings of shrubs, subdivision of shrubs and banana tree graphs. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi and Selvagopal in [8]. Maryati et al. [11] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h. The (b,d)-H-antimagic labeling was introduced by Inayah et al. [6]. Inayah et al. in [6], [7] introduced (b,d)-H-antimagic labeling for some shackles of a connected graph H.

The present paper investigated the super $(b,1)$-S_{p+2}-antimagic labeling of Sun graphs SG_n^p with p pendant edges.

2. Star antimagicness of Sun graphs

A star graph S_n is a tree consisting of one vertex adjacent to n vertices. In other words, a complete bipartite graph $K_{1,n}$ is called a Star S_n.

The Sun graph SG_n^p, $n \geq 3, p \geq 1$, is constructed from a cycle C_n by inserting p pendant edges with every vertex of the C_n. The vertices and edges on the cycle will be called the *cycle vertices* and the *cycle edges* respectively. The remaining edges will be termed as the *pendant edges* and their end points as *pendant vertices*. The Sun Graph SG_n^p contains $n(p+1)$ vertices and edges.

$$V(SG_n^p) = \{x_i, y_i^j : 1 \leq j \leq p\}, \text{ and}$$

$$E(SG_n^p) = \{x_i x_{i+1}, x_i y_i^j : 1 \leq j \leq p\},$$

where indices i are taken modulo n.

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2.1 Super S_{p+2}-antimagic Labeling of Sun Graph SG_n^p

Let S_{p+2} be a star on $p+3$ vertices. Every star $S_{p+2}^k, k=1,2,\ldots,n$ in SG_n^p has the vertex set

$$V(S_{p+2}^{(k)}) = \{x_{k-1}, x_k, x_{k+1}\} \cup \{y_k^j : 1 \le j \le p\}$$

and the edge set

$$E(S_{p+2}^{(k)}) = \{x_{k-1}x_k, x_k x_{k+1}, x_k y_k^j : 1 \le j \le p\},$$

where indices are taken modulo n.
Under a total labeling α, the $S_{p+2}^{(k)}$-weights are:

$$\begin{aligned} wt_\alpha(S_{p+2}^{(k)}) &= \sum_{v \in V(S_{p+2}^{(k)})} \alpha(v) + \sum_{e \in E(S_{p+2}^{(k)})} \alpha(e). \\ &= \sum_{s=k-1}^{k} (\alpha(x_s) + \alpha(x_s x_{s+1})) + \alpha(x_{k+1}) + \sum_{s=1}^{p} (\alpha(y_k^s) + \alpha(y_k^s x_k)) \\ &= Sum_1 + Sum_2 \end{aligned} \tag{1}$$

where

$$Sum_1 = \sum_{s=k-1}^{k} (\alpha(x_s) + \alpha(x_s x_{s+1})) + \alpha(x_{k+1}) \tag{2}$$

$$Sum_2 = \sum_{s=1}^{p} (\alpha(y_k^s) + \alpha(y_k^s x_k)) \tag{3}$$

Theorem 2.1. *The Sun graph SG_n^p admits a super $(b,1)$-S_{p+2}-antimagic labeling where $n \ge 3$, $p \ge 1$ are positive integers and S_{p+2} is a star on $p+3$ vertices.*

Proof. Proof is divided into two parts:
In the first part, we give a P_3-antimagic labeling of cycle C_n. In second part, we show that weights of pendant edges with P_3-antimagic labeling of cycle C_n gives us super $(b,1)$-S_{p+2}-antimagic labeling.
The total labeling α for cycle C_n is defined as:

$$\alpha(x_i) = i$$
$$\alpha(x_i x_{i+1}) = 2n(p+1) + 1 - i$$

where indices are taken modulo n. Under labeling α, cycle vertices are labeled with $\{1,2,\ldots,n\}$ and cycle edges are labeled with $\{n(2p+1)+1, 2, \ldots, 2n(p+1)\}$.
Using (2) and above labeling, the sum for $P_3^{(k)}$-weights are:

$$\begin{aligned} Sum_1 &= \alpha(x_{k-1}) + \alpha(x_k) + \alpha(x_{k+1}) + \alpha(x_{k-1}x_k) + \alpha(x_k x_{k+1}) \\ &= 3k + 4n(p+1) + 3 - 2k \\ &= 4n(p+1) + 3 + k \end{aligned} \tag{4}$$

which shows a super P_3-antimagic labeling of cycle C_n.

For $j = 1,2,\ldots,p$ and $j \equiv 1 \pmod 2$, the labeling of pendent edges and their end vertices is defined as:

$$\{\alpha(y_i^j), \alpha(x_i y_i^j)\} = \{(j+1)n + 1 - i, (p+j)n + i\}$$

For $j = 1,2,\ldots,p$, $j \equiv 0 \pmod 2$

$$\{\alpha(y_i^j), \alpha(x_i y_i^j)\} = \{jn + i, (p+j+1)n + 1 - i\}$$

where the smallest possible labels $\{n+1, n+2, \ldots, n(p+1)\}$ appear on the end vertices of pendant edges and the pendant edges receive labels $\{n(p+1)+1, n(p+1)+1+2, \ldots, n(2p+1)\}$. Therefore α is total labeling of SG_n^p.
Also, for $j = 1,2,\ldots,p$

$$\alpha(y_i^j) + \alpha(x_i y_i^j) = n(p + 2j + 1) + 1 \tag{5}$$

Using (3) and (5), we have

$$Sum_2 = \sum_{s=1}^{p} \big(\alpha(y_{(k,s)}) + \alpha(x_k y_{(k,s)})\big) = p[2n(p+1)+1] \tag{6}$$

Equation (1), (4) and (6) gives

$$\begin{aligned} wt_\alpha(S^{(k)}_{p+2}) &= 4n(p+1)+3+k+p[2n(p+1)+1]. \\ &= 2n(p+1)(p+2)+p+3+k \end{aligned}$$

which consists of consecutive integers with initial term $a = 2n(p+1)(p+2)+p+4$. This completes the proof.

□

2.2 $S_3(r)$ antimagic labeling of r-Subdivided Sun graph $SG_n^{(1)}(r)$

Let $G(r)$ be a graph (denoted as r-subdivided graph of G) obtained from the graph G by inserting $r \geq 1$ new vertices into every edge of G.

Let $SG_n^1(r)$ be r-subdivided graph of Sun graph SG_n^1 with $2n(r+1)$ vertices and edges.

$$V(SG_n^1(r)) := \{x_{(i,j)}, y_{(i,j)} : 1 \leq i \leq n, 0 \leq j \leq r\}$$

$$E(SG_n^1(r)) := \{x_{(i,j)}x_{(i,j+1)}, y_{(i,j)}y_{(i,j+1)} : 0 \leq j \leq r-1\} \cup \{x_{(i,r)}x_{(i+1,0)}, y_{(i,r)}x_{(i,0)}\}$$

with indices are taken modulo n.

Let $S := S_3(r)$ be r-subdivided graph of star graph S_3 with $3(r+1)+1$ vertices and $3(r+1)$ edges.

The k^{th} r-subdivided star $S^{(k)}$ has the vertex set

$$V(S^{(k)}) = \{x_{(k-1,0)}, x_{(k,0)}, x_{(k+1,0)}\}, \{y_{(k,j)} : 0 \leq j \leq r\}$$

and the edge set as follows:

$E(S^{(k)}) = \{x_{(k-1,j)}x_{(k-1,j+1)}, x_{(k,j)}x_{(k,j+1)}, y_{(k,j)}y_{(k,j+1)} : 0 \leq j \leq r-1\} \cup \{y_{(k,r)}x_{(k,0)}, x_{(k,r)}x_{(k+1,0)}\}$, where indices are taken modulo n.

Under a total labeling β, the $wt_\beta(S) := wt(S_3(r))$ are:

$$\begin{aligned} wt_\beta(S^{(k)}) &= \sum_{v \in V(S^{(k)})} \beta(v) + \sum_{e \in E(S^{(k)})} \beta(e). \\ &= \sum_{s=k-1}^{k+1} \beta(x_{(s,0)}) + \sum_{s=k-1}^{k} \sum_{j=0}^{r-1} \big(\beta(x_{(s,j)})\beta(x_{(s,j+1)})\big) + \sum_{s=k-1}^{k} \beta(x_{(s,r)}x_{(s+1,0)}) \\ &+ \sum_{j=0}^{r} \beta(y_{(k,j)}) + \sum_{j=0}^{r-1} \beta(y_{(k,j)})\beta(y_{(k,j+1)}) + \beta(y_{(k,r)}x_{(k,0)}) \\ &= Sum_1 + Sum_2 \end{aligned} \tag{7}$$

where

$$Sum_1 = \sum_{s=k-1}^{k+1} \beta(x_{(s,0)}) + \sum_{s=k-1}^{k} \sum_{j=0}^{r-1} \big(\beta(x_{(s,j)})\beta(x_{(s,j+1)})\big) + \sum_{s=k-1}^{k} \beta(x_{(s,r)}x_{(s+1,0)}) \tag{8}$$

$$Sum_2 = \sum_{j=0}^{r} \beta(y_{(k,j)}) + + \sum_{j=0}^{r-1} \beta(y_{(k,j)})\beta(y_{(k,j+1)}) + \beta(y_{(k,r)}x_{(k,0)}) \tag{9}$$

Theorem 2.2. *The Sun graph $SG_n(r)$ admits a super $(b,1)$-S-antimagic labeling where $n \geq 3$, $r \geq 1$ are positive integers and S is an r-subdivided graph of S_3. .*

Proof. Firstly, we give a $P_3(r)$-antimagic labeling of r-subdivide cycle $C_n(r)$. The total labeling β for r-subdivided cycle $C_n(r)$ is defined as:

$$\begin{aligned} \beta(x_{(i,j)}) &= nj + i & j &= 0,1,2,\ldots,r \\ \beta(x_{(i,j)}x_{(i,j+1)}) &= n(4r+4-j)+1-i & j &= 0,1,2,\ldots,r-1 \\ \beta(x_{(i,r)}x_{(i+1,0)}) &= n(4r+4-r)+1-i & j &= 0,1,2,\ldots,r-1 \end{aligned}$$

where indices are taken modulo n. Under labeling β, vertices of r-subdivided cycle are labeled with $\{1, 2, \dots, n(r+1)\}$ and edges of r-subdivided cycle are labeled with $\{3n(r+1)+1, 3n(r+1)+2, \dots, 4n(r+1)\}$.

Using (8) and above labeling, the sum for $P_3^{(k)}(r)$-weights are:

$$\begin{aligned}\sum_{s=k-1}^{k+1}\beta(x_{(s,0)}) + \sum_{s=k-1}^{k}\sum_{j=0}^{r-1}\big(\beta(x_{(s,j)})\beta(x_{(s,j+1)})\big) + \sum_{s=k-1}^{k}\beta(x_{(s,r)}x_{(s+1,0)}) &= (r+1)\{4n(r+1)+1\}+i \\ &= (r+1)(4nr+4n+1)+i \qquad (10)\end{aligned}$$

which shows a super $P_3(r)$-antimagic labeling of r-subdivided cycle $C_n(r)$.
Labeling of pendent edges and their end points is defined as:

$$\begin{aligned}\beta(y_i^j) &= n(2r+2-j)+1-i && j = 0, 1, 2, \dots, r \\ \beta(y_i^j y_{(i,j+1)}) &= n(2r+2+j)+i && j = 0, 1, 2, \dots, r-1 \\ \beta(y_{(i,r)}x_{(i,0)}) &= n(2r+2-r)+i && j = 0, 1, 2, \dots, r-1\end{aligned}$$

where the smallest possible labels $\{n(r+1)+1, n(r+1)+2, \dots, 2n(r+1)\}$ appear on the end points of pendant edges and the pendant edges receive labels $\{2n(r+1)+1, 2n(r+1)+2, \dots, 3n(r+1)\}$. Therefore β is total labeling of SG_n^p.

$$\begin{aligned}\sum_{j=0}^{r}\beta(y_{(k,j)}) + \sum_{j=0}^{r-1}\beta(y_{(k,j)})\beta(y_{(k,j+1)}) + \beta(y_{(k,r)}x_{(k,0)}) &= (r+1)\{4n(r+1)+1\} \\ &= 4n(r+1)^2 + (r+1) \qquad (11)\end{aligned}$$

Using equations (7), (10) and (11), we get

$$\begin{aligned}wt_\beta(S^{(k)}) &= 8n(r+1)^2 + 2(r+1)] + i \\ &= 2(r+1) + (4nr+4n+1) + i\end{aligned}$$

which constitute a sequence of consecutive integers with initial term
$a = 8n(r+1)^2 + 2(r+1)] + i$. This completes the proof.

□

3. Conflict of Interests

The author(s) declare that there is no conflict of interests.

4. Acknowledgements

The authors are grateful for the valuable comments of the anonymous referees.

References

[1] Chen, Lian, Huiqin Jiang, Zehui Shao, and M. Ivanovic. "Dominator and total dominator colorings in vague graphs." Eng. Appl. Sci. Lett. 2, no. 2 (2019): 10-17.
[2] W. Gao., Remarks on Fractional Locally Harmonious Coloring., Open J. Math. Sci., Vol. 2 (2018), Issue 1, pp. 301 - 306.
[3] W. Gao, Fakhir Aslam, Zohaib Zahid, Sohail Zafar., 3-total edge mean cordial labeling of some standard graphs., Open J. Math. Sci., Vol. 3 (2019), Issue 1, pp. 129 - 138.
[4] A. Gutiérrez and A. Lladó, Magic coverings, *J. Combin. Math. Combin. Comput.* **55** (2005), 43–56.
[5] N. Inayah, A. Lladó and J. Moragas, Magic and antimagic H-decompositions. *Discrete Math.*, **312** (2012), 1367–1371.
[6] N. Inayah, A.N.M. Salman and R. Simanjuntak, On (a, d)-H-antimagic coverings of graphs, *J. Combin. Math. Combin. Comput.* **71** (2009), 273–281.
[7] N. Inayah, R. Simanjuntak, A.N.M. Salman and K.I.A. Syuhada, On (a, d)-H-antimagic total labelings for shackles of a connected graph H, *Australasian J. Combin.* **57** (2013), 127–138.
[8] P. Jeyanthi and P. Selvagopal, More classes of H-supermagic graphs, *Internat. J. Algor. Comput. Math.* **3** (2010), 93–108.
[9] A. Lladó and J. Moragas, Cycle-magic graphs, *Discrete Math.* **307** (2007), 2925–2933.
[10] T.K. Maryati, E.T. Baskoro and A.N.M. Salman, P_h-(super)magic labelings of some trees, *J. Combin. Math. Combin. Comput.* **65** (2008), 198–204.
[11] T.K. Maryati, A.N.M. Salman and E.T. Baskoro, Supermagic coverings of the disjoint union of graphs and amalgamations, *Discrete Math.* **313** (2013), 397–405.
[12] H. M. Nagesh, M. C. Mahesh Kumar., Block Digraph of a Directed Graph., Open J. Math. Sci., Vol. 2 (2018), Issue 1, pp. 202 - 208.
[13] A.A.G. Ngurah, A.N.M. Salman and L. Susilowati, H-supermagic labelings of graphs, *Discrete Math.* **310** (2010), 1293–1300.
[14] H. M. Nagesh & M. M. Kumar, Block digraph of a directed graph. Open J. Math. Sci., 2(3) (2018), 202-208.
[15] Shah, Aqsa, S. A. U. H. Lingua, and S. A. U. H. Bokhary. "On chromatic polynomial of certain families of dendrimer graphs." Open Journal of Mathematical Sciences 3.1 (2019): 404-416.
[16] Yan, L., MOHAMMAD REZA Farahani, and W. Gao. "Distance-based indices computation of symmetry molecular structures." Open Journal of Mathematical Sciences 2, no. 1 (2018): 323-337.